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A statistical equilibrium representation of markets as complex networks

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Abstract

We represent an exchange economy in terms of statistical ensembles for complex networks by introducing the concept of market configuration. In this way, starting from economic reasoning, we obtain a sound interpretation of the typical network variables in terms of thermodynamic quantities together with a strong consistency with microeconomic theory, and in particular with Walrasian general equilibrium theory. In our formalism, naturally arises the interpretation of the temperature T as a quantification of economic disequilibrium, which can indeed coexist with statistical equilibrium.

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1 Introduction

The first attempt to draw a link between classical thermodynamics and economics, due to Samuelson [1], was based on the parallel between entropy maximization and utility or profit maximization. Under this perspective, economic agents are treated like thermodynamic macroscopic subsystems composing a larger macroscopic system which is the market [2]. This view, although consistent with the supposed optimizing behavior of economic agents, is at odds with the statistical physics microfoundation of thermodynamics, according to which macroscopic equilibrium arises from a purely random behavior of micro units. Statistical physics introduces a clear separation of the microscopic and macroscopic levels which is lost in the proposed economic parallelism.

A recent stream of literature employs the techniques of statistical mechanics under the hypothesis that in a limited period of time an economic system may behave as though in a quasi-equilibrium state [3, 17]. In these models we have homogeneous agents randomly exchanging money with a constraint on the total amount of money M in the system. In particular in [17] statistical ensembles describing economic systems with money (cash) and credit-debt are defined in an analytic way without resorting to numerical simulations. The homogeneity of economic agents is a consequence of the assumption of uniform (in physical terms, isotropic) random exchange, since the latter hypothesis makes agents symmetric with respect to the probability distribution of their monetary holdings.

In this paper we attempt to take a step forward by describing with the tools of statistical physics an economic system with heterogeneous agents. We resort to complex networks theory in order to introduce the key notion of market configuration. In [18, 19] the authors consider a set of network realizations (ensemble) and then impose constraints on the expectation value of a given set of graph observables $\{x_i\}$ with respect to the ensemble. This approach, as we see in the next section, lends itself to a very natural interpretation in terms of economic equilibrium. On the other hand, this is not the usual approach of statistical mechanics, where the constraint is imposed on the total 'Hamiltonian' of the system. In practice, the ensemble introduced in [18] is obtained, with the standard methods, only in the specific

case $T = 1$, provided that the energy levels are appropriately chosen. By introducing a generic value of the temperature T, it is possible to regain the whole formalism of statistical mechanics, and consequently a consistent microfoundation of thermodynamic variables and relationships. It should be noticed that for a generic value of T we have statistical but not economic equilibrium. The equality between the two is regained at $T = 1$ (or for some other fixed value T_0, see note 4), provided that the energy levels in the two Hamiltonians are equal. Hence the principal motivation of this paper is to define ensembles for complex networks with a sound interpretation of the typical network variables in terms of thermodynamics quantities together with a strong consistency with economic theory in general, and with Walrasian general equilibrium theory in particular. In the literature, the important concept of graph temperature has been introduced for the first time in [20] within a grand canonical ensemble. Moreover, in [21] the temperature of a complex network is studied in terms of the clustering properties of the graph.

The structure of the paper is the following. In section 2 we discuss the main ideas to build the ensembles of complex networks. In sections 3,4,5 we build the ensembles. In section 6 we apply our modeling to illustrate the role played by T in terms of statistical uncertainty over market states. In section 7 we further analyze the parallel between thermodynamic and economic equilibrium as addressed in the economic literature. Section 8 is devoted to some conclusions.

2 Preliminaries

In microeconomic theory an exchange economy is a system of N consumers, initially endowed with ω_i units of the available M commodities and with preferences over alternative price-dependent consumption vectors $x_i(p) \in B_i$, where B_i (the *budget set*) is a set of affordable consumption vectors for i, given ω_i and p. In this economy, trade occurs whenever agents prefer a consumption vector $x_i \neq \omega_i$. The excess demand function $z_i = x_i - \omega_i$ describes the trading plan of each agent.

A *Walrasian* equilibrium occurs when the market-clearing condition is satisfied for some price vector p^*:

$$\sum_i z_i(p^*) = 0 \tag{1}$$

and at the same time the consumption vector of each individual is the preferred one x_i^* in her own budget set. The existence of the equilibrium price

vector p^*, which can be proved invoking Katutani's fixed-point theorem, is regarded as a fundamental achievement of microeconomic theory [22].

The notion of Arrow-Debreu equilibrium extends Walrasian equilibrium by introducing the notions of states of the world $s = 1, \ldots, S$ and of contingent commodities, i.e. of commodities whose delivery is conditional on the realized state of the world. Then an Arrow-Debreu equilibrium is simply a Walrasian equilibrium in which the individual consumption vector x_i is a $M \times S$ vector. With this formulation, x and ω become dependent on s. In fact, they are random variables distributed according to the discrete probability $P(s)$. The Arrow-Debreu equilibrium requires that each individual formulates a complete set of preferences over state-dependent consumption vectors $x(s)$. Additionally, it requires that a market exists for each contingent commodity. These *forward* markets open before the uncertainty over the state of the world is resolved. Once the latter is revealed, commodities are traded according to the corresponding equilibrium price and consumption vectors. There is no motivation to open a *spot* market for re-trading, because forward markets achieve for some state s exactly the same outcome that is achieved by a spot market opened when all participants know that the state of the world is s. Finding the Arrow-Debreu equilibrium price vector p^* is thus equivalent to finding the state-dependent Walrasian equilibrium price vector $p^*(s)$ for each $s = 1, \ldots, S$.

According to a well known result of microeconomic theory, the requirement of a complete set of forward markets can be dropped. In fact, it is possible to show that the equilibrium achieved by trading on a forward market for a single commodity alongside with spot markets for all commodities (the so-called Radner Equilibrium) is equivalent to an Arrow-Debreu equilibrium under the condition that agents know the market-clearing prices for all commodities and under each state.

Surprising as it may be, a statistical representation of markets arises naturally from standard microeconomic theory. In order to illustrate this point, we need to introduce the notion of market configuration, defined as a sequence of nonnegative values $\{w_{ij}\}$ describing the flow of a given commodity from i to j, which can be arranged in a nonnegative matrix W. With this in mind, we introduce the following notations: $x_i = \sum_j w_{ji}$ is the final allocation for agent i, and the excess demand is defined as follows

$$z_i = \sum_{j \neq i} (w_{ji} - w_{ij}) = x_i - \sum_j w_{ij}. \tag{2}$$

Thus we obtain that $\omega_i = \sum_j w_{ij}$, where it is understood that w_{ii} stands for the fraction of the initial endowment which is not exchanged on the

market. As we explain in sec. 3, a Walrasian equilibrium corresponds to many distinct market configurations, which is natural to consider as random and equiprobable since economic agents are indifferent between them. This observation opens the door to a statistical representation of market configurations by means of the microcanonical ensemble.

Starting from this background, the transition to complex networks theory is very natural. In fact, the representation of markets outlined above is nothing different from the matrix representation of a network G made of N vertices [1]. Thanks to this parallelism, we can take advantage of previous contributions from the complex networks literature. In the paper [18], the authors adapt the powerful tools of statistical mechanics to the study of random networks. In order to grasp the importance of this approach for our representation of markets, we need in the first place to distinguish between statistical equilibrium and economic equilibrium. Indeed, statistical equilibrium in the sense of statistical physics, is a more general concept, meaning that we can define a time independent probability measure over market configurations $P(G)$. If the system is allowed to relax without external disturbances, it will converge to $P(G)$ for $t \to \infty$, but the achievement of a Walrasian equilibrium cannot be taken for granted unless some additional arrangements are introduced.

It is possible to show that, following the method of Park and Newman[18], we may obtain economic equilibrium as a consequence of statistical equilibrium. The rationale of their approach is straightforward, since they seek a systematic way to generate ensembles of random graphs displaying a set of desired average properties $\{x_i\}$. In practice, they consider a set of graph configurations $G \in \Im$ imposing the constraint that the expectation value of graph observables $\{x_i\}$ with respect to the ensemble probability distribution $P(G)$ is equal to some arbitrary value $\bar{x}_i$. $P(G)$ is obtained maximizing the Gibbs entropy S

$$S = -\sum_{G \in \Im} P(G) \ln P(G), \tag{3}$$

together with the observables and normalization constraints

$$\langle x_i \rangle = \sum_{G \in \Im} P(G) x_i(G) = \bar{x}_i, \quad \sum_{G \in \Im} P(G) = 1. \tag{4}$$

[1] For this reason, in this paper the terms "market" and "network" are used as equivalents even if, strictly speaking, a market is a directed weighted network. If G is a binary network, its links can take only binary values, and thus its matrix representation is given by the matrix A with binary entries. For convenience, in this paper binary and weighted networks are labeled respectively fermionic and bosonic networks by analogy with the terminology of statistical physics.

As a result, they obtain the distribution

$$P(G) = \frac{e^{-H(G)}}{Z}, \quad Z = \sum_{G \in \Im} e^{-H(G)}, \quad H(G) = \sum_i \theta_i x_i(G), \tag{5}$$

where $\{\theta_i\}$ are Lagrange multipliers[2].

The degree and strength distributions are the most common observables chosen as constraints for the problem (3)-(4). By strength of a node i in a bosonic symmetric network we define the sum $w_i = \sum_j w_{ij}$. If W is asymmetric, i.e. G is directed, we need to distinguish between the out-strength w_i^{out} and in-strength w_i^{in}. The degree of a node i, instead, is defined over the binary matrix A representation of a fermionic network as the sum $k_i = \sum_j a_{ij}$. If A is asymmetric, again we need to distinguish between the out-degree and in-degree of the node i. Park and Newman provide a solution of the problem (3)-(4) when the constraints are represented either by the degree or by the strength distributions. They took into account the following Hamiltonian for a fermionic undirected network with fixed expected degree distribution:

$$H = \sum_i k_i \theta_i. \tag{6}$$

Using the definition of k_i eq. (6) may be rewritten as follows

$$H = \sum_{i<j} \epsilon_{ij} a_{ij}, \tag{7}$$

where $\epsilon_{ij} = \theta_i + \theta_j$. For a bosonic network they obtain instead

$$H = \sum_{i<j} \epsilon_{ij} w_{ij}. \tag{8}$$

Starting from the expressions (7)-(8), they derive the analogue of the quantum Fermi-Dirac and Bose-Einstein distributions respectively for $\langle a_{ij} \rangle$ and $\langle w_{ij} \rangle$. By inspection it is easy we see that the w_{ij} and a_{ij} are indeed equivalent to the occupation numbers included in the Hamiltonian of the Fermi and Bose ideal gases respectively, while the ϵ_{ij} are equivalent to the energy levels occurring in the same Hamiltonian. This equivalence justifies

[2]There is a bijective relationship between a given network model and the corresponding statistical ensemble, since solving a given model means actually to find the particular equilibrium distribution which is consistent with the constraints of that model. For this reason the terms "model" and "market ensemble" or "network ensemble" can be used as equivalents.

the interpretation, made by the authors, according to which links are the equivalent of particles in network ensembles.

In economic terms, links / particles are commodity units while the energies ϵ_{ij} are very naturally interpreted as shadow prices [23]. Furthermore, given our previous market representation, in the bosonic directed case it is possible to make the straightforward identifications:

$$\omega_i = w_i^{out} \tag{9}$$

$$x_i = w_i^{in} \tag{10}$$

$$z_i = w_i^{in} - w_i^{out} \tag{11}$$

These identifications make sense only if $\omega_i(s)$ and $x_i(s)$ have negligible fluctuations around their ensemble averages. In fact, the problem (3)-(4) is related to the derivation of the canonical ensemble in statistical physics, which requires that fluctuations of constrained observables are small. From our arguments above, we see that this is not necessarily the case for Arrow-Debreu equilibria, since $P(s)$ is arbitrary. On the other hand, microeconomic theory makes the unrealistic assumption that we have a complete knowledge of the world, represented by the finiteness of the set of states. If we suppose instead that there is a numerable infinite set of states, we can compute $x^*(s)$ and $p^*(s)$ for at most a finite sample of states S. But then the equivalence between forward markets and spot markets, which is essential for microeconomic theory, comes to depend critically on the shape of $P(s)$. Only if the latter is sharply peaked around the most likely state $\hat{s}$, this equivalence may be reformulated in the following form:

$$\langle \omega_i \rangle \approx \omega_i(\hat{s}) \approx \omega_i(s) \tag{12}$$

$$\langle p^* \rangle \approx p^*(\hat{s}) \approx p^*(s) \tag{13}$$

$$\langle x_i^* \rangle \approx x_i^*(\hat{s}, p^*) \approx x_i^*(s, p^*) \tag{14}$$

where s is the realized state. When this condition is fulfilled, we say that the market is in a *statistical* Walrasian equilibrium. We label this equilibrium as Walrasian because agents' uncertainty regarding the final allocation is strongly limited under our assumptions, and the computation of p^* is necessary in order to achieve the equilibrium itself. The conditions above are satisfied as soon as we choose the following constraints for the problem (3)-(4):

$$\left\langle w_i^{out} \right\rangle = \omega_i(\hat{s}) \tag{15}$$

$$\left\langle w_i^{in} \right\rangle = x_i^*(\hat{s}, p^*) \tag{16}$$

In fact, we know from statistical physics that the fluctuations around the expected values are small in the resulting network ensemble for large $L = \sum_i \omega_i$. With this choice we justify the previous claim that economic equilibrium can be obtained from statistical equilibrium in the bosonic directed model under appropriate conditions[3].

We should note two fundamental differences of the distribution (5) with respect to the usual Boltzmann-Gibbs distribution. In the first place, the former is coincident with the latter only for $T = 1$. Since T is related to the energy constraint, we see that H is not explicitly conserved in the network ensemble, but its conservation is a consequence of the constraints over observables. Thus, it easy to see that if we allow T to vary we cannot comply anymore with the constraints of the model[4].

In the following sections, we build the fermionic and bosonic network ensembles along standard lines [3], thereby recovering the factor $1/T$ of statistical physics. In order to do so, we introduce the generic energy levels ϵ_{ij} together with the Hamiltonian

$$H(G) = \sum_{ij} \epsilon_{ij} \sigma_{ij}. \tag{17}$$

Of course, it is always possible to specify the energy levels in such a way as to verify the constraints (15)-(16) for $T = 1$ or some other fixed value (see note 4). In this particular case, our conditions for statistical Walrasian equilibrium are regained under the more general model. This possibility suggests that T could be interpreted in this context as a quantification of economic disequilibrium, which can indeed coexist with statistical equilibrium. In other terms, statistical equilibrium should be interpreted as the outcome of a relaxation process which not necessarily leads the system to economic equilibrium.

For the derivations of the following sections, we propose to identify the equivalent of the volume V of physical systems for networks with the degrees of freedom of the network. These are given by the number of points of the bidimensional lattice formed by the couples of nodes (i, j), i.e. $V = O(N^2)$. For example, for directed graphs we have $V = N(N - 1)$, for undirected

[3]In principle, it is possible to provide an economic interpretation also for degree distributions. In fact, the latter represent the number of transaction made by agents. If these transaction are costly, it makes sense for agent to economize and fix a desired level for them. We will not pursue this identification in this paper, but we will focus instead on the economic interpretation of bosonic networks.

[4]Of course, it is always possible to make the substitution $\epsilon_{ij} = \epsilon'_{ij}/T_0$ for some T_0. In this case the constraints are satisfied only for $T = T_0$.

ones $V = N(N-1)/2$ while for graphs with selfloops $V = N^2$. Moreover, according with the usual interpretation (see [18, 19]), the number of links, that we denote from now on with L, is the formal analogue of the number particles in the usual physical systems. In the economic interpretation, the fluctuations of L, introduced with the grandcanonical ensemble, are a consequence of uncertainty over the available commodities under different states of the world, since $L = \sum_i \omega_i(s)$. With this assumptions and ideas in mind, in the next three sections we derive the statistical ensembles.

3 Microcanonical ensemble

Let's suppose that agents are certain about the state of the world $\bar{s}$, i.e. $P(x) = \delta(x - \bar{s})$. Then, provided they know the market clearing price p^*, they are free to choose their optimal consumption x^* and the market will display with certainty an "energy" level $E = \sum_i (\lambda_i x_i^* + \theta_i \omega_i)$. Here the λ_i, θ_i are nothing more than multipliers for the following constraints:

$$w_i^{out} = \omega_i(\bar{s}) \tag{18}$$

$$w_i^{in} = x_i^*(\bar{s}) \tag{19}$$

In general there are many market configurations which are consistent with these constraints. Since agents have no reason to prefer one of this states over the other, the latter can be considered as equally likely. In this way we obtain the microcanonical ensemble, i.e. the set of market configurations which are consistent with Walrasian equilibrium.
The starting point to build the microcanonical ensemble is the Hamiltonian (17)[5] with $\epsilon_{ij} = \lambda_i + \theta_j$. In this ensemble, the "energy" $H(G) = E$ is conserved and the number of links or commodity units L and of agents N are fixed too. By considering these constraints, as usual, we can define $\Gamma(E)$ as the total number of configurations, at fixed $V = O(N^2)$ and L, calculated at the surface $H(G) = E = const.$ of constant energy. In this context, the entropy S can be defined in the usual way:

$$S(E, V, L) = \ln \Gamma(E, V, L). \tag{20}$$

[5]From a more general point of view we can attach to any vertex i micro-variables $\{x_i^1, ...x_i^k\}$. In the most general form we have $H_i(G) = f(x_i^1, .., x_i^k)$ with $H(G) = \sum_i H_i(G)$.

The following thermodynamical relations hold:

$$dS = \frac{\partial S}{\partial E} dE + \frac{\partial S}{\partial V} dV, \tag{21}$$

$$\left(\frac{\partial S}{\partial V}\right)_{E,L} = \frac{P}{T}, \quad \left(\frac{\partial S}{\partial E}\right)_{V,L} = \frac{1}{T}, \tag{22}$$

where P is the pressure definition in the network context. Furthermore, we define the analogue of the free Helmholtz energy F:

$$F = E - TS, \tag{23}$$

$$dF = dE - TdS - SdT \tag{24}$$

From (21) and (22) we get:

$$TdS = dE + PdV. \tag{25}$$

Equation (25) is the analogue of the first thermodynamic principle for complex networks. Finally, from (24) and (25) we have:

$$dF = -PdV - SdT, \tag{26}$$

$$\left(\frac{\partial F}{\partial V}\right)_{T,L} = -P, \quad \left(\frac{\partial F}{\partial T}\right)_{V,L} = -S. \tag{27}$$

4 Canonical ensemble

Now let's suppose that the probability distribution of states is peaked around a given state $\bar{s}$ and that x^*, ω and p^* are continuous in s. In this case the conditions for a statistical Walrasian equilibrium, specified in sec. 2, are fulfilled. Agents trade according to the observed $s \neq \bar{s}$, but the probability of observing large deviations from Walrasian equilibrium, as defined by the microcanonical ensemble, is small.

Inspired by this economic view, the derivation of the canonical ensemble follows the lines present in [3, 24] , whereby we follow as closely as possible the standard derivation of the Gibbs distribution in statistical mechanics. As a first step we introduce G_1 and G_2 as sub-networks of an isolated network G of fixed energy E with $E_1 \ll E_2$. Then we assume that $H(G) \simeq H(G_1) + H(G_2)$, which means that the states (i, j), defined between couples of nodes i, j belonging respectively to G_1 and G_2, are empty, i.e. that links or commodity units cannot flow from one subnetwork to the

other. Thus the two systems can be treated as mutually independent. Under the assumption that E is conserved, we have

$$\Gamma(E) \simeq \Gamma_1(E_1)\Gamma_2(E_2 = E - E_1). \tag{28}$$

As usual, by performing a taylor expansion with $E_1 \ll E$, since we are interested in the behavior of the subsystem 1 independently from the reservoir 2 we can write, after using (20) and (22),

$$Z = \sum_G e^{-\frac{H(G)}{T}}, \quad P(G) = Ce^{-\frac{H(G)}{T}}, \;\; C = \frac{1}{Z}. \tag{29}$$

Following the usual derivation in statistical mechanics we can write the partition function

$$Z = e^{-\frac{F}{T}}. \tag{30}$$

As stated above, differently from (5), the standard factor $\frac{1}{T}$ appears in (29)-(30). The two distributions coincide for $T = 1$ provided that the energy levels in the two Hamiltonians are identical.

5 Grand canonical ensemble

Since the Park and Newman distribution is a special case of the standard Gibbs distribution, the grand canonical extension may be accomplished along the standard lines[6]. Here it is essential to stress again that the number N is related to the volume of the network, i.e. $V = O(N^2)$ is the equivalent of the volume V of physical systems. We suppose that G_1 and G_2 are components of a larger system G characterized by the Gibbs distribution (29), but now links are allowed to "move" between the two volumes V_1 and V_2. The partition function Z of G is

$$Z_G(L, V, T) = \sum_G \exp\left(-\frac{H(G)}{T}\right) = \tag{31}$$

$$= \sum_{L_1=0}^{L} \sum_{G_1} \exp\left(-\frac{H(G_1)}{T}\right) \sum_{G_2} \exp\left(-\frac{H(G_2)}{T}\right), \tag{32}$$

[6]We observe that Park and Newman don't derive explicitly the grand canonical partition function, although they use it implicitly to solve their model. Even if they don't introduce the chemical potential, their results are correct since, as underlined in [19], the latter can be always absorbed in the energy terms ϵ_{ij}.

Also in this case, by following the derivation of the grand canonical partition function of the standard statistical mechanics [24] (remember that in our case L plays the role of particle numbers N in a ordinary gas), by the help of (29) and (30) after posing $\mu = \left(\frac{\partial F}{\partial L}\right)_{V,T}$, $P = -\left(\frac{\partial F}{\partial V}\right)_{L,T}$ and introducing the fugacity $z = \exp\left(\frac{\mu}{T}\right)$, after a taylor expansion, we obtain the grand canonical partition function:

$$\mathcal{Q}(z, V, T) \equiv \sum_{L=0}^{\infty} z^L Z_G(L, V, T), \tag{33}$$

where the subscript '1' has been dropped in (33). Moreover, the standard relationships still hold:

$$\log \mathcal{Q} = \frac{PV}{T} \tag{34}$$

$$\langle L \rangle = z \, \frac{\partial}{\partial \, z} \log \mathcal{Q}. \tag{35}$$

Furthermore, we have

$$dS = \frac{dE}{T} + \frac{P}{T} dV - \frac{\mu}{T} dL, \tag{36}$$

$$dF(T, V, L) = -PdV - SdT + \mu dL, \tag{37}$$

and

$$TdS = dE + PdV - \mu dL. \tag{38}$$

The analogue of the Gibbs-Duhem relation gives

$$SdT = VdP - Ld\mu. \tag{39}$$

It should be noticed that our approach allows us to study the result of a variation of the temperature for complex graphs. We define the specific heat C_V by

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,L}. \tag{40}$$

Discontinuities of C_V are related to phase transitions. Let's see how to compute $\mathcal{Q}$ in the case of a directed fermionic network. Since $L = \sum_{i \neq j} \sigma_{ij}$,

and thanks to (17), the grand partition function reads:

$$\mathcal{Q}(z,V,T) = \sum_{L=0}^{n(n-1)} z^L Z_G(L,V,T) = \sum_{\{\sigma_{ij}\}} \exp\left[\frac{\mu L - H}{T}\right] \tag{41}$$

$$= \sum_{\{\sigma_{ij}\}} \exp\left(\sum_{i\neq j} \frac{(\mu-\epsilon_{ij})\sigma_{ij}}{T}\right) = \tag{42}$$

$$= \prod_{i\neq j}\left[1+\exp\left(\frac{\mu-\epsilon_{ij}}{T}\right)\right]. \tag{43}$$

The expected occupation numbers $\langle\sigma_{ij}\rangle$ are obtained in the usual way

$$\langle\sigma_{ij}\rangle = -T\frac{\partial}{\partial\epsilon_{ij}}\log\mathcal{Q} \tag{44}$$

$$= \frac{\exp\left(\frac{\mu-\epsilon_{ij}}{T}\right)}{1+\exp\left(\frac{\mu-\epsilon_{ij}}{T}\right)}, \tag{45}$$

which coincides with the results of Park & Newman for $T=1$.
In the bosonic case, instead of (43) we have

$$\mathcal{Q}(z,V,T) = \prod_{i\neq j}\left[\frac{1}{1-\exp\left(\frac{\mu-\epsilon_{ij}}{T}\right)}\right]. \tag{46}$$

Espression (46) it gives

$$\langle\sigma_{ij}\rangle = \frac{1}{\exp\left(\frac{-\mu+\epsilon_{ij}}{T}\right)-1}, \tag{47}$$

that is the Bose-Einstein distribution for the network. In this way we can also regain the results of [19] but in a more general context since all the variables needed to build thermodynamics (P,T,V,L,μ) are defined.
The expected value of a product of m links in general is given by

$$\langle\sigma_{ij}\ldots\sigma_{hk}\rangle = (-T)^m\frac{1}{\mathcal{Q}}\frac{\partial}{\partial\epsilon_{ij}\ldots\partial\epsilon_{hk}}\mathcal{Q}$$

We also obtain the usual results for fluctuations of σ_{ij}:

$$\langle \sigma_{ij}^2 \rangle - \langle \sigma_{ij} \rangle^2 = -T \frac{\partial F}{\partial^2 \epsilon_{ij}} = T^2 \frac{\partial \log \mathcal{Q}}{\partial^2 \epsilon_{ij}} = \frac{\exp\left(\frac{\mu - \epsilon_{ij}}{T}\right)}{\left[1 \pm \exp\left(\frac{\mu - \epsilon_{ij}}{T}\right)\right]^2} \tag{48}$$

In general the covariance of σ_{ij} and σ_{hk}, where either $i \neq h$ or $j \neq k$, is

$$\langle \sigma_{ij} \sigma_{hk} \rangle - \langle \sigma_{ij} \rangle \langle \sigma_{hk} \rangle = -T \frac{\partial F}{\partial \epsilon_{ij} \partial \epsilon_{hk}} = 0 \tag{49}$$

Of course, this result can be obtained directly from the fact that σ_{ij} and σ_{hk} are independent. The degrees or strengths w_i instead are positively correlated [19]. Using $\epsilon_{ij} = \theta_i + \theta_j$ we obtain

$$\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = -T \frac{\partial F}{\partial \theta_i \partial \theta_j} = \begin{cases} \dfrac{\exp\left(\frac{\mu - \epsilon_{ij}}{T}\right)}{\left[1 \pm \exp\left(\frac{\mu - \epsilon_{ij}}{T}\right)\right]^2} & \text{for } i \neq j \\ \sum_j \dfrac{\exp\left(\frac{\mu - \epsilon_{ij}}{T}\right)}{\left[1 \pm \exp\left(\frac{\mu - \epsilon_{ij}}{T}\right)\right]^2} & \text{for } i = j \end{cases} \tag{50}$$

Once again both results can be derived just by assuming that the σ_{ij} are independent. In particular for $i \neq j$ we have the following

$$\langle \sigma_i \sigma_j \rangle = \left\langle \sum_k \sum_{k'} \sigma_{ik} \sigma_{jk'} \right\rangle \tag{51}$$

$$= \sum_k \sum_{k'} \left\langle \sigma_{ik} \sigma_{jk'} \right\rangle = \langle \sigma_{ij}^2 \rangle + \sum_{k \neq i} \sum_{k' \neq j} \langle \sigma_{ik} \rangle \langle \sigma_{jk'} \rangle \tag{52}$$

With an equivalent argument we can prove that, in a directed network with statistically independent links, out-degrees (in-degrees) and out-strengths (in-strengths) are reciprocally uncorrelated, while they are positively correlated with in-degrees (out-degrees) and in-strengths (out-strengths).

6 An application: Graph Thermodynamics

In this section we apply the statistical ensembles presented in the sections above, and in particular the grand canonical one, to study the role of the temperature in our model. To this purpose, we want to consider simple

expressions for the energies ϵ_{ij}.
To start with, the grand canonical partition function $\mathcal{Q}$ defined in (43) can be rewritten as:

$$\mathcal{Q} = \sum_A \exp(\mu L_A - H_A)/T, \tag{53}$$

where the pedix A denotes the graph elements $\{\sigma_{ij}\}$ and thus the probability of a graph A is given by:

$$P_A = \frac{1}{Z} \exp(\mu L_A - H_A)/T \tag{54}$$

The energy (hamiltonian) function of a graph is given by eq. (17):

$$H_A \equiv H = \sum_{ij} \epsilon_{ij}\sigma_{ij} \tag{55}$$

where σ_{ij} is the adjacency matrix. Without loss of generality we take into account the case of a fermionic network, i.e. $\sigma_{ij} = 0, 1$. In the following we reproduce the results obtained in [20] for the two limiting cases $T \to \infty$ and $T \to 0$. We begin by studying the limit for $T \to \infty$:

$$\lim_{T\to\infty} Z = 2^{N(N-1)/2} \tag{56}$$

for an undirected graph. Thus in the limit $T \to \infty$ all the graphs in the ensemble have the save probability:

$$P_A = 2^{-N(N-1)/2}, \qquad \forall A \tag{57}$$

In order to derive an analytical result for the low temperature limit, we have to specify an ansatz for the energies ϵ_{ij}. The most simple ansatz is given by $\epsilon_{ij} = \epsilon$ for each i, j. In this case it is straightforward to derive the limit $T \to 0$. We denote by A the graph with the maximum allowed L_A (with N fixed). The probability can be written as:

$$P_A = \frac{1}{\sum_{B\neq A} \exp\left[(\mu - \epsilon)(L_B - L_A)/T\right] + 1} \tag{58}$$

By assuming $\mu > \epsilon$ we get:

$$\lim_{T\to 0} P_A = 1 \tag{59}$$

since each single term in the summation goes to zero for $L_B < L_A$. Moreover $P_B \equiv 0 \ \forall B \neq A$. The graph with the highest probability is the complete connected one (see fig. 1). In the case $\mu < \epsilon$ the only graph which survives

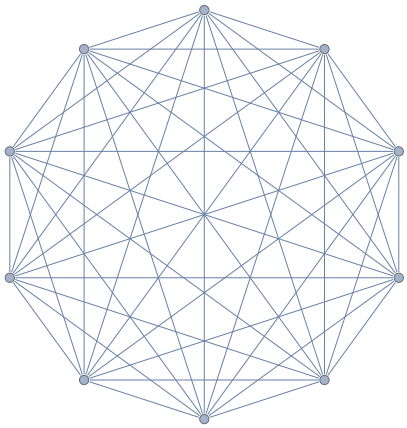

Figure 1: Totallly connected graph with $N = 10$. This graph has probability $P_A = 1$ in the limit $T \to 0$ for $\mu > \epsilon$.

is the totally disconnected one, i.e $L_A = 0$ with probability:

$$P_A = \frac{1}{\sum_{B \neq 0} \exp\left[(\mu - \epsilon)\, L_B / T\right] + 1} \tag{60}$$

that in the limit $T \to 0$ gives:

$$\lim_{T \to 0} P_A = 1 \tag{61}$$

In the following there are results for ϵ_{ij} random variables with given distribution. In particular from now on we assume ϵ_{ij} gaussian iid variables with $\bar{\epsilon_{ij}} = 1$ and standard deviation $\sigma = 0.5$. With the previous assumptions we cover in particular the case $\epsilon_{ij} = \epsilon_i + \epsilon_j$ as well as more general functional dependences. In figure 2 there is the energy distribution, i.e. number of graph with a given energy, for a graph ensemble with $N = 10$, $\mu = 10$, $T = 10^4$. This distribution simply reflects the different combinatorial factors for a graph with fixed N and a variable number of edges. In particular there is only one way to get the totally connected graph, while for example there are 2769 ways to get a topologically inequivalent graph with $N = L = 10$.

Having in mind the economic interpretation of a graph ensemble, it is worth studying how the probability of a given graph changes with the temperature T. In particular we want to compute the number of graphs with a given probability. This quantity is crucial since each (inequivalent) realization of a graph $G = \{N, L\}$ *represents a possible configuration of the market*.

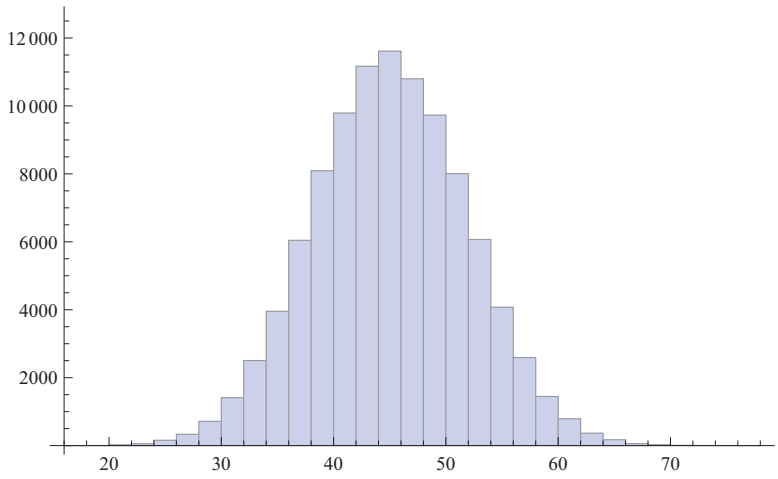

Figure 2: Energy distribution with ϵ_{ij} random variables with gaussian distribution.

Let us start from the two formal limits previously analytically derived, $T \to 0$ and $T \to \infty$, under the assumption $\mu > \epsilon$, i.e. the non trivial case:

- $T \to 0$, only the totally connected graph survives with $P_A = 1$. This means that we have *no uncertainty* on the market configuration s although we are, economicly speaking, far from a Walrasian equilibrium since the latter is not consistent with any variation of T.

- $T \to \infty$, all the graphs are equally likely with probability $P_A = 2^{-N(N-1)/2}$ (undirected fermionic graph), i.e. we have the maximum uncertainty on the market configuration.

Let us now what happens for T interpolating between these two exact results. We will rely on some numerical random graph sampling. The sampling dimension is 10^5. The results for $T = 20$, 50, 100 are encoded in the plots 3 showing for a fixed temperature the number of graphs as a function of their associated probability. The probability is the grandcanonical one (fixed number of vertices N and variable numebr of edges L) written in eq. (54). It is possibile to see that in the first two cases that the vast majority (10^5) of graphs have negligible probability, while for $T = 10^2$ the probability starts spreading more "democratically" among the graphs. These behaviour is even more clear taking into account higher temperature values (see plot 4. For these higher temperatures the "collapse" of all the

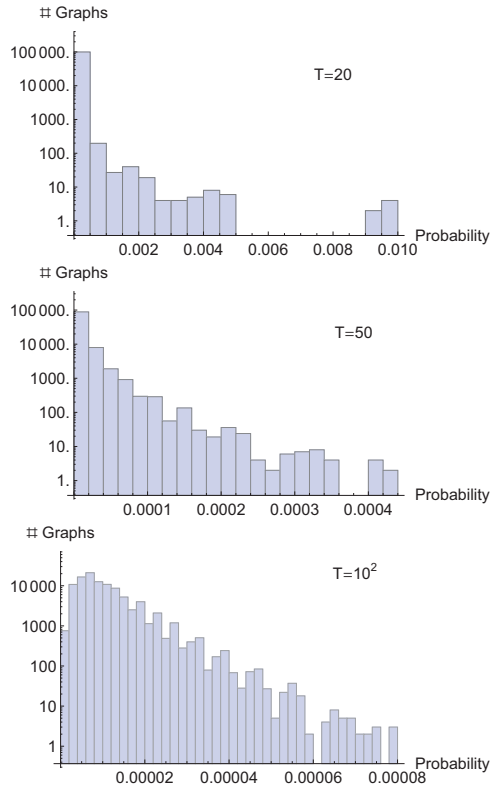

Figure 3: Number of graphs with given probability for $T = 20$, 50, 100.

graphs towards the probability 10^{-5} (that allowed by the finite sampling) is well rendered. It is remarkable that these numerical results are robust. In fact the same behaviour holds even for a more "realistic" economic setup, namely ϵ_{ij} gaussian distributed. Moreover the same results still hold in the case of a bosonic network, where of course the limit $T \to 0$ is associated to a condensed state with particles occupying the state or states with the lowest energy. However this is not surprising since the bosonic case can be obtained from the fermionic one by simply rescaling the energies.
As a further consideration, it is interesting to see how thermodynamics transformations can act to change the value of T. In the usual thermodynamics, the temperature variation of a given system can be obtained by a change in the volume, where for example a given gas is enclosed, by keeping fixed the other thermodynamical quantities, or by an energy exchange with a given reservoir. By analogy, in a complex network we may think to a transfor-

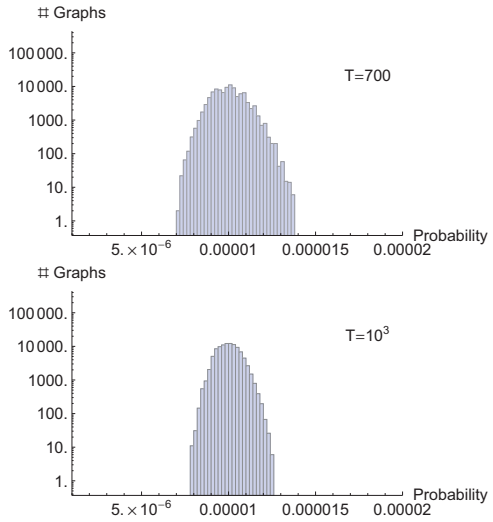

Figure 4: Number of graphs with given probability for $T = 7 \cdot 10^2$, 10^3.

mation changing the vertex number N (i.e. the volume). As an example, if vertices represent a system of banks, then this can be accomplished by a merger of two or more banks or by the failure of some bank. Another possibility can be energy exchange. By eq. (17), this can be obtained by a change in the number of links L or by a change in the values of the ϵ_{ij} . A variation of L, which represents a net inflow or outflow of commodities, has a clear interpretation in terms of open economic systems. Concerning the change in the energies ϵ_{ij}, this can have an interesting economic interpretation. In fact, by the influence of some external economic input, the agents can change strategy or "behaviour" and thus they can change the energy distributions ϵ_{ij} by reaching another equilibrium state with a different T. Note that this generally can be accomplished out of a Walrasian equilibrium. However, we may think also to a change of the energy levels in such a way that the system evolves through states of Walrasian equilibrium. In practice, a mismatch between economic and statistical equilibrium can arise when a market is subject to an unexpected external shock, that in our formalism is described by a variation of T which is not compensated by an adjustment of the energy levels. We may think instead that under an external shock, the latter are generally very different from the starting ones. But, if the flow of external shocks is fast enough to prevent the relaxation of the system or the adjustment of expectations, economic equilibrium becomes

impossible. Conversely, for a very short relaxation time, economic equilibrium can be reached. The reasonings above show that our formalism allows to explore in a sound way the meaning of all the relevant thermodynamical transformations.

7 Discussion

Samuelson [25] expressed the view that no more than a formal mathematical analogy exists between classical thermodynamics and utility theory. This analogy is grounded in the mathematical tool of constrained optimization, which is applied respectively to entropy and to the individual utility function. More recently, Smith and Foley have elaborated further on this parallel [2]. It is interesting to compare their approach with the one we present in this paper. From a methodological point of view, they are quite different: Smith and Foley leverage on the parallel between utility and entropy, and consequently try to mutually adapt neoclassical utility theory and classical thermodynamics; we leverage instead on the indeterminacy of Walrasian equilibrium with respect to market configurations in order to bring statistical physics and thermodynamics into general equilibrium theory, without any explicit reference to utility.

The main motivation for Smith and Foley to push the parallel between utility and entropy is to lend support to a decentralized process of price discovery, which can be contrasted with Walrasian tâtonnement [26]. On the contrary, we use Walrasian equilibrium as a starting point to introduce all thermodynamic quantities, including entropy, with a clear microfoundation in terms of market configurations.

Smith and Foley explicitly choose to neglect the separation between micro and macro by treating economic agents in an exchange economy as thermodynamic macroscopic systems. In particular, they draw a parallel between the process of discovering market-clearing prices in such economy and the partial equilibration of a collection of thermodynamic systems. The mental experiment they propose, in its simplest form, is the following: two isolated thermodynamic systems with different extensive and intensive variables are brought into contact in such a way that they are thermodynamically closed one with respect to the other. Then the system formed by the two subsystems will equilibrate up to a unique value of the intensive variables (temperature and pressure) while, if the systems are of different chemical nature, the extensive variables will generally be different. In the economic interpretation, relative prices represent the intensive variables, and

the individual consumption vectors represent the extensive variables of the two subsystems. While thermodynamic equilibrium is obtained by maximizing entropy $S(E, V)$ as a function of the extensive variables, economic equilibrium is obtained by maximizing $U(X)$ as a function of the vector of available commodities X.

Their parallel between entropy and utility runs into severe difficulties. In the first place, the value of entropy for a macroscopic system at equilibrium is a state function of the system itself, with a well defined value. This is not true for economic systems, because the maximum of $U(X)$ is degenerate unless we suppose that agents exchange commodities at the market clearing prices. In this case we obtain a Walrasian equilibrium, which becomes unique under some additional conditions. But Walrasian equilibrium in this context is at odds with thermodynamics since if the intensive variables (relative prices) are already equal, no change in the values of extensive variables (and thus no market exchange) is possible from a thermodynamic perspective. In the second place, the idea that economic agents are thermodynamic systems implies that each agent can maximize her own utility in isolation, something which is completely at odds with utility theory. In the third place, entropy is additive, while utility has the same property only under restrictive assumptions (quasi-linear utility functions). In the fourth place, economic equilibration in the neoclassical theory is subject to the voluntary exchange principle, which implies the additional constraint that individual utilities can never decrease. This constraint prescribes that no individual agent will end up with a consumption vector which is smaller than the endowment vector, while this is a perfectly possible outcome in thermodynamics[7]. Last but not least, the notion of system in thermodynamics is intentionally undetermined, since systems may be not only be brought into contact but mixed to form a new larger system whose original components cannot be distinguished anymore. But this is at odds with the very nature of economic agents.

8 Conclusions

In this paper we have extend the approach started in [18] and developed in [20] in order to define in a comprehensive way the thermodynamics of complex networks. Furthermore, we have provided a clear economic inter-

[7]The simplest example is a system made of two subsystems of fixed volumes and different temperatures: the initially hotter subsystem has lower internal energy and equal volume at equilibrium.

pretation of our results, showing that statistical equilibrium can be defined also for systems which are not in economic equilibrium. In this context, the temperature T for a given network can be seen as a measure of the departure from economic equilibrium.

In the same way of physics, the relevance of statistical equilibrium for networks must be traced back to the dynamics of relaxation. Concerning the exchange economy introduced in sec. 2, the relaxation process can be represented as follows. At t_0 the market configuration is specified by $w_{ij} = \delta_{ij}\omega_i$. If we suppose that all agents know each other's excess demand, then it is rational for each agent i to choose for her transactions the counterpart j which minimizes $f = |z_i + z_j|$. Then the quantity $\min(|z_i|, |z_j|)$ is exchanged. When $z = \sum_i |z_i|$ is minimized, there is no more incentive to trade and the system is in a rest state. In order to obtain economic equilibrium, the system must achieve $z = 0$. In both cases, the system is driven towards a stable state which is the one described in the previous sections.

As an application, we have considered different simple expressions for the "energies" ϵ_{ij}. In particular, we have considered the simple cases with constant and gaussian distributed ϵ_{ij}, and studied the ensemble probability of a given graph in terms of the temperature T. What we obtain is that the number of graphs with a given probability seems to be practically insensitive to the choice made for ϵ_{ij} (see also [20]).

It should be stressed that our approach differs from the most common parallelism between utility theory and thermodynamics [7, 2]. In fact, we do not identify the utility function with entropy and maintain instead the fundamental distinction between micro and macro. Conversely, we start from general equilibrium theory by introducing the equiprobable Walrasian equilibrium states as the building blocks of a statistical representation of market configurations. This provides a natural link with complex networks, and in particular a clear connection between graph and thermodynamic variables which preserves the core of microeconomic theory.

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