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State Space Model to Detect Cycles in Heterogeneous Agents Models

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Abstract

We propose an empirical test to depict possible endogenous cycles within Heterogeneous Agent Models (HAMs). We consider a 2-type HAM into a standard small-scale dynamic asset pricing framework. On the one hand, fundamentalists base their expectations on the deviation of fundamental value from market price expecting a convergence between them. On the other hand, chartists, subject to self-fulling moods, consider the level of past prices and relate it to the fundamental value acting as contrarians. These pricing strategies, by their nature, cannot be directly observed but can cause the response of the observed data. For this reason, we consider the agents' beliefs as unobserved state components from which, through a state space model formulation, the heterogeneity of fundamentalist-chartist trader cycles can be mathematically derived and empirically tested. The model is estimated using the S&P500 index, for the period 1990-2020 at different time scales, specifically, daily, monthly, and quarterly.

Key words: Heterogeneous Agents Models, Endogenous Cycles, State Space Model, Kalman Filter

JEL codes: C13, G10, E32

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1 Introduction

In his seminal paper, Kirman (1992) shows that the choice of one 'representative' standard utility-maximizing individual 'is not simply an analytical convenience [...], but is both unjustified and leads to conclusions which are usually misleading and often wrong'. In this line, the heterogeneous agents models (HAMs) reveal how complex dynamics of price fluctuations are related to the interactions among (groups) of agents. This behavioral approach undermines two important theoretical pillars: market efficiency based on rational behavior, and homogeneous investors. The first one is replaced with the idea that economic agents have limited cognitive abilities. The agents, unable to observe all the information about the state of the economy, form their beliefs following regressive and extrapolative expectations feeding the boom and bust in real and financial cycles. The second one is substituted with the idea that heterogeneity among agents is pervasive.

HAMs' theoretical literature is mature and the results achieved are strong (see as survey Hommes, 2006; LeBaron, 2006; Franke, 2008; Chiarella et al. 2009). On the other hand, there is a growing empirical literature, even if we cannot observe any consensus on the estimation methodology, nor conclusive results (for recent survey refers to Lux and Zwinkel 2018; Ter Ellen and Verschoor, 2018). Franke and Westerhoff (2017) note two approaches: direct and indirect. The first method employs surveys to measure the sentiments of a specific group of the population, typically the momentum traders, and thus explain their behavior. The second considers a model as a whole and strives to estimate all its parameters in one effort. Concerning the latter, we can distinguish between two types of inference methods (Kukacka and Barunik, 2017). In the first, key structural features of HAMs can be estimated in a straight way. Depending on the complexity of the models, we can list - among other - the nonlinear least squares, the maximum and quasi-maximum likelihood among others. In the second method, estimation based on simulating artificial data from the model is used instead: through the method of simulated moments it is possible to depict phenomena which are consequence of behavioral biases, such as volatility clustering and long-memory effects.

In general, these works confirm the theoretical models, revealing that sentiment dynamics, are important in explaining stylized facts observed in financial time series and in replicating observed anomalies in financial markets. Moreover, price movements in asset prices can derive from a switching mechanism that moves agents among different rules: an evolutionary competition generates price fluctuations that may be triggered by differences in trading rules employed and amplified by dynamics between different strategies.

However, the papers are different both in methods, time-scale and data used. Boswijk et al. (2007) estimate the Brock and Hommes (1998) model through nonlinear least squares (AR1 process), using Annual US stock prices between 1871 and 2003. Estimating belief coefficients and the intensity of choice, they produce evidence of the existence of both fundamentalists and chartists. In the same line Hommes and in't Veld (2017), using quarterly data of S&P500 for the period 1950-2016, compare the HAM model with a representative agent model, showing that a switching regime amplifies fluctuations. Franke (2009) applies the Method of Simulated Moment to a HAM model with the aim to compare the moments computed simulating the model and those present in exchange rates as well as stock market indexes. In the same line, Franke and Westeroff (2011), using daily data for S&P500, assess the overall fit of the model (belief coefficients, the transition probability approach, and the discrete choice approach), finding that the predisposition towards chartism and the discrete choice approach better fit the data.

Alfarano et al. (2006), use a maximum likelihood approach on Australian stock (1980-2004) and foreign exchange data (1983-2004). They assess the empirical relevance of regime switching probabilities among fundamentalists and noise traders strategies in a herding mechanism. The results indicate the dominance of fundamentalist activity even if the hypothesis of a symmetric setting cannot be statistically rejected. Lof (2012) uses a smooth-transition autoregressive (STAR) model, where strategies depend on macroeconomic fluctuations, in order to capture under which conditions either chartists or fundamentalists prevail within the market. He finds that chartists dominate during an economic boom, while fundamentalists otherwise. Chiarella et al. (2012) use a HAM model within a Markov switching framework for the monthly data of S&P500 in the first decade of the century. They find that heterogeneity is pervasive also within groups and reveal the transition between boom and bust periods. Chiarella et al. (2014) use the maximum likelihood estimation, adapting the Chiarella and He (2003) model. They confirm the presence of both chartists and fundamentalists in the stock market (i.e. S&P 500), where the formers are destabilizing and the latter stabilizing force. Kukacka and Barunik (2017) formally test an HAM à la Brock and Hommes (1998) trough a non parametric simulated maximum likelihood estimation for six stock market indices. Overall they find a dominance of agents who adopt trend follower beliefs in absence of a switching process.

It is worth noting, the papers mentioned above, as well as our paper, refer primarily to the heterogeneous agents models; the heterogeneity concerns the groups - specifically the strategies adopted. It is worth underlining that a great effort also concerns the econometric validation of agent-based models, which we will call *pure*, in which heterogeneity is pervasive, given that it is at the level of the single agent. As well highlighted by LeBaron and Tesfatsion (2008) and by Grazzini and Richiardi (2015), the estimation of pure agent models is complicated by the heterogeneity of the agents but also by the difficulty of aggregating their actions. Recently, Delli Gatti and Grazzini (2020) propose both a Bayesian procedure to estimate the parameters of a pure agent model and a method to aggregate the data obtained from the simulations so to obtain historical series to be used in the model validation phase.

In a recent empirical paper, Lux (2018) uses Markov Chain Monte Carlo methods for State Space Models. He emphasizes that this methodology could be optimal for HAMs precisely because they typically have latent components and observable variables. In the same line, Gusella and Stockhammer (2021) propose an empirical test for Minskyan financial cycles in asset prices, driven by the presence of dogmatic fundamentalists and momentum traders. They use the Kalman filter to identify the two behavioral rules and evaluate whether the conditions for cycles hold.

Our paper departs from Gusella and Stockhammer (2021) to depict possible endogenous cycles, but highlighting the heterogeneity among agents and seeking to empirically identify different evaluation behaviors. On the one hand, we do not consider *dogmatic* fundamentalist behavior. In our case, fundamentalists base their expectations on the deviation of fundamental value from market price, expecting a convergence between them. The fundamental value follows either a random walk as in Franke (2008), or it is identified through the Gordon model (1959), as in Chiarella et al. (2012). On the other hand, chartists, subject to self-fulling moods, consider the level of past prices and relates it to the fundamental value acting as contrarians[1]. Additionally, studying our model at different frequency times, we make a structural break analysis.

Differently from Lux (2018), our quotas are fixed and we consider not observable the strategies used. Moreover, we focus our attention on cycles. We also believe that a missing point in this literature is an analysis of the model at different temporal span. Often in theoretical models, little attention is paid to what t is, time; and those who did empirical studies used different time scales. We believe that models may not explain series trends for all time scales. We, therefore, dedicate a focus to the implementation of our model at different time scales, specifically, daily, monthly, quarterly.

To our best knowledge, the empirical HAMs do not provide evidence of cycles emerging directly from the data as a consequence of behavioral heuristics and we do believe that state space model has not been thoroughly investigated for these types of problems. Therefore, the present paper considers and estimates a 2-type heterogeneous agent model into a standard small-scale dynamic asset pricing framework to provide an empirical test for endogenous financial cycles that emerge from the two latent expectations rules. In other words, we know that unobservable variable(s) drive(s) the dynamics of the observable time series; we test directly if this dynamics is a cyclical dynamic in the observed asset prices.

To depict the possible endogenous cycles, we propose the following estimation strategy. We consider the agents' beliefs as unobserved state components from which, through a state space model formulation, the endogeneity of fundamentalist-contrarians trader cycles can be mathematically derived. Once we obtain the cyclical conditions, maximum likelihood estimation via the Kalman filter over the state space model is performed. In this way, we can evaluate the presence of endogenous cycles directly from the data and investigate whether they are statistically significant or not. Besides the tractability of the model, the main advantage of this framework is that, through filtering information on unobserved states, it is able to test whether behavioral rules lead to the cyclical dynamics in the observed asset prices. Moreover, we estimate the relative shares of the two economic agents in the market with the possibility to

[1]Throughout the entire paper we will use chartists, speculators and contrarians interchangeably.

recover the parameters associated with the two expectation rules to assess the degree of chartist and mean-reverting coefficients. After the estimation, the iterative Kalman filter algorithm is used to extract the unobserved states by performing forward recursion over the state-space model.

The rest of the paper is organized as follows. In section (2) we discuss the methodology employed and the empirical strategy. Main results are reported in section (3). Specifically, we show firstly those when the fundamental value follows a random walk in (3.1) and later when the fundamental value is derived from the Gordon model (3.2). Section (4) reports our conclusions and final considerations.

2 Methodology

This section describes the proposed modeling strategy to study the presence of endogenous financial fluctuations due to heterogeneous groups of agents in the financial asset market.

The asset price P at time t is determined by the following equation:

$$P_t = P_t^f + B_t^h\left(P_{t-1}^f, P_{t-1}\right)$$

where P_t^f is the fundamental value of the asset, $B_t^h\left(P_{t-1}^f, P_{t-1}\right)$ is the function which describes how agents of type h form their beliefs. The arguments in function B are the prices that have been observed in the recent past. We consider two types of agents: fundamentalists and contrarians. In this way, the belief function can be defined as follow:

$$B_t^h\left(P_{t-1}^f, P_{t-1}\right) = \gamma B_t^f\left(P_{t-1}^f, P_{t-1}\right) + (1-\gamma)\, B_t^c\left(P_{t-1}^f, P_{t-1}\right) \tag{1}$$

where the weights γ and $1-\gamma$ are the proportions of fundamentalists and chartists respectively. Using (1), the asset price can be rewrite as follow

$$P_t = P_t^f + \gamma B_t^f\left(P_{t-1}^f, P_{t-1}\right) + (1-\gamma)\, B_t^c\left(P_{t-1}^f, P_{t-1}\right) \tag{2}$$

The belief functions can be specified as the difference between the expected asset price of the agents of type h and the base price P^b, which is the price that enter in the formalization of their heuristic expectations (on this point see Franke, 2009):

$$B_t^h = E_t^h\left(P_{t+1}\right) - P_{t-1}^b$$

The base price can be the fundamental value at time $t-1$ $\left(P_{t-1}^f\right)$ or the current asset price at time $t-1$ $\left(P_{t-1}\right)$.

As a consequence of this, the structure of price expectations can have two different forms:

$$E_t^h\left(P_{t+1}\right) = P_{t-1} + B^h\left(P_{t-1}^f, P_{t-1}\right) \qquad \left(P_{t-1}^b = P_{t-1}\right) \tag{3}$$

or

$$E_t^h\left(P_{t+1}\right) = P_{t-1}^f + B^h\left(P_{t-1}^f, P_{t-1}\right) \qquad \left(P_{t-1}^b = P_{t-1}^f\right) \tag{4}$$

Let us consider the fundamentalist traders. The fundamentalists believe that the asset price may temporarily deviate from the fundamental value, ${P_t}^f$. However, they also believe that the price will eventually converge to the fundamental value. Their demand for asset price is proportional to the difference between the market price and the fundamental value. So the fundamentalists' expectation can be defined as a weighted average of the most recent price and the fundamental price:

$$E_t^f\left(P_{t+1}\right) = \tilde{\lambda} P_{t-1}^f + \left(1-\tilde{\lambda}\right) P_{t-1} \qquad \left(0 \leq \tilde{\lambda} \leq 1\right) \tag{5}$$

where $\tilde{\lambda}$ measures the speed of mean-reversion of the market price to the fundamental value. This means that, in the case of asset price boom or bust, fundamentalists expect market prices to revert to the fundamental value. In the HAMs literature (Franke, 2008), P_t^f, the fundamental value, often follows a random walk process:

$${P_t}^f = P_{t-1}^f + \varepsilon_t \qquad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_\varepsilon^2\right) \tag{6}$$

where ε_t is the individual disturbance term which is normally distributed with mean zero and variance σ_ε^2. Later, in a slightly modification of the model, we will relax the assumption of the random walk using the Gordon growth model (Gordon, 1959).

With respect to the belief function B^f and referring to (3) and (4), the expectation formation can be rewritten as:

$$B_t^f = \lambda\left(P_{t-1}^f - P_{t-1}\right) \tag{7}$$

If the base price is equal to the current asset price at time before ($P_{t-1}^b = P_{t-1}$), with respect to (5) we have that $\lambda = \tilde{\lambda}$ with $0 \leq \lambda \leq 1$. If the base price is equal to the fundamental price at time before ($P_{t-1}^b = P_{t-1}^f$), with respect to (5), we have that $\lambda = \tilde{\lambda} - 1$ with $-1 \leq \lambda \leq 0$.

As to the chartists, we define their expectation behavior in the following way:

$$E_t^c\left(P_{t+1}\right) = \tilde{\beta} P_{t-1}^f + \left(1-\tilde{\beta}\right) P_{t-1} \qquad \left(\tilde{\beta} \leq 0\right) \tag{8}$$

Chartists consider the level of past prices and relate it to the fundamental value acting as contrarians. With $\beta < 0$, when the asset price is above (below) its fundamental value, it follows that the economic agent optimistically (pessimistically) believes in a further price increase (decrease). This form of expectation can be defined as a form of speculation on the difference between the current price trend and the fundamental value.

Given B^c and referring to (3) and (4), respectively, the expectation formation can be rewritten as:

$$B_t^c = \beta \left(P_{t-1}^f - P_{t-1} \right) \tag{9}$$

As before, if the base price is equal to the observed asset price at time before ($P_{t-1}^b = P_{t-1}$), with respect to (8) we have that $\beta = \tilde{\beta}$ with $\beta \leq 0$. If the base price is equal to the fundamental price at time before ($P_{t-1}^b = P_{t-1}^f$), with respect to (8), we have that $\beta = \tilde{\beta} - 1$ with $\beta \leq -1$. We do not impose a specific base price. Of course, the base price must be the same for chartists and fundamentalists.

As agents' behavioral beliefs are unobserved variables, we can construct our state space model in the context of the unobserved components model. Substituting (2) in (7), the belief function for fundamentalist traders can be rewritten as:

$$\begin{aligned} B_t^f &= \lambda \left[P_{t-1}^f - \left(P_{t-1}^f + \gamma B_{t-1}^f + (1-\gamma) B_{t-1}^c \right) \right] \\ &= -\lambda \gamma B_{t-1}^f - \lambda (1-\gamma) B_{t-1}^c \end{aligned} \tag{10}$$

While, substituting (2) in (9), the belief function for chartists can be rewritten in the following way:

$$\begin{aligned} B_t^c &= \beta \left[\left(P_{t-1}^f - P_{t-1}^f - \gamma B_{t-1}^f - (1-\gamma) B_{t-1}^c \right) \right] \\ &= -\beta \gamma B_{t-1}^f - \beta (1-\gamma) B_{t-1}^c \end{aligned} \tag{11}$$

We set

$$\begin{cases} a_{11} = -\lambda \gamma \\ a_{12} = -\lambda (1-\gamma) \\ a_{21} = -\beta \gamma \\ a_{22} = -\beta (1-\gamma) \end{cases} \tag{12}$$

such that, in a stochastic form, we finally obtain:

$$B_t^f = a_{11} B_{t-1}^f + a_{12} B_{t-1}^c + \varphi_t \qquad \varphi_t \sim \mathrm{N} \left(0, \sigma_\varphi^2 \right) \tag{13}$$

$$B_t^c = a_{21} B_{t-1}^f + a_{22} B_{t-1}^c + \eta_t \qquad \eta_t \sim \mathrm{N}\left(0, \sigma_\eta^2\right) \tag{14}$$

where φ_t and η_t are the individual disturbance term which are normally distributed with mean zero and variance σ_φ^2 and σ_η^2 respectively.

Eq. (6) with Eqs. (13) and (14) are the so-called state equations. Together with the observed asset price (2), they represent our state space model. An essential feature of any state space model is that the state equation must be a first-order stochastic difference equation (Enders, 2016). In our model, the observed equation of the state space model is:

$$P_t = \begin{pmatrix} \gamma & 1-\gamma & 1 \end{pmatrix} \begin{pmatrix} B_t^f \\ B_t^c \\ {P_t}^f \end{pmatrix}$$

Taking into account Eqs. (6), (13), and (14) we have the transition equation of the state space model:

$$\begin{pmatrix} B_t^f \\ B_t^c \\ {P_t}^f \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_{t-1}^f \\ B_{t-1}^c \\ P_{t-1}^f \end{pmatrix} + \begin{pmatrix} \varphi_t \\ \eta_t \\ \varepsilon_t \end{pmatrix}$$

which it can be rewritten in compact form as:

$$P_t = H Z_t$$

$$Z_t = A Z_{t-1} + \delta_t \qquad \delta_t \sim \mathrm{N}\left(0, Q\right)$$

where P_t is the observable asset price,

$$Z_t = \begin{pmatrix} B_t^f \\ B_t^c \\ {P_t}^f \end{pmatrix}$$

is the state vector,

$$H = \begin{pmatrix} \gamma & 1-\gamma & 1 \end{pmatrix}$$

is the measurement matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the transition matrix and δ_t is the vector containing the state disturbance of unobserved components, normally distributed with mean zero and variances collected in the diagonal matrix Q.

With this modeling strategy, we can reveal the nature and the cause of the dynamic movement of observed variables in an effective way. Indeed, with a state space model, it is possible to explain the behavior of an observed variable by examining the internal dynamic properties of the unobserved components. In other words, we can analyze the effect of the unobserved components variable (heuristic beliefs) on the observed variable (asset prices).

The dynamics of the system is given by the transition equation which describes the evolution of the vector of unknown latent variables. Eigenvalues analysis can be performed to study the conditions for oscillations in our discrete dynamic system associated with the unobserved components. We obtain the associated characteristic equation considering the following determinant of the transition matrix:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0$$

so that

$$(1 - \lambda) \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

First of all, we have the following eigenvalue:

$$\lambda_3 = 1$$

In addition, regarding the other two eigenvalues, they must satisfy:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

from which

$$a_{11}a_{22} - \lambda a_{11} - \lambda a_{22} + \lambda^2 - a_{12}a_{21} = 0$$

$$\lambda^2 + \lambda\left(-a_{11} - a_{22}\right) + \left(a_{11}a_{22} - a_{12}a_{21}\right) = 0$$

$$\lambda_{1,2} = \frac{-\left(-a_{11} - a_{22}\right) \pm \sqrt{\left(-a_{11} - a_{22}\right)^2 - 4\left(a_{11}a_{22} - a_{12}a_{21}\right)}}{2}$$

In order to have an oscillating behavior, these two last eigenvalues have to be complex, so that we require:

$$\Delta = \left(-a_{11} - a_{22}\right)^2 - 4\left(a_{11}a_{22} - a_{12}a_{21}\right) < 0$$

i.e.:

$$a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0 \tag{15}$$

When this is the case:

$$\lambda_{1,2} = \frac{-\left(-a_{11} - a_{22}\right)}{2} \pm i\frac{\sqrt{-\left[\left(-a_{11} - a_{22}\right)^2 - 4\left(a_{11}a_{22} - a_{12}a_{21}\right)\right]}}{2} = a + ib$$

The complex number in the cartesian form $a \pm ib$ can be written in the equivalent trigonometric form $\rho\left(\cos\omega \pm i\sin\omega\right)$.

In order to have oscillations of constant amplitude, we require:

$$\rho = \left(a^2 + b^2\right)^{\frac{1}{2}} = 1$$

where ρ is the modulus of the complex number.

Solving this simple equation with respect to the parameters of interest, we obtain:

$$\sqrt{\frac{\left(-a_{11} - a_{22}\right)^2}{4} + \frac{-\left(-a_{11} - a_{22}\right)^2 + 4\left(a_{11}a_{22} - a_{12}a_{21}\right)}{4}} = 1$$

i.e.:

$$a_{11}a_{22} - a_{12}a_{21} = 1$$

If the condition in Eq. (15) holds, with $a_{11}a_{22} - a_{12}a_{21} < 1$ ($\rho < 1$) we have damped oscillations. With $a_{11}a_{22} - a_{12}a_{21} > 1$ ($\rho > 1$) we have explosive oscillations.

As stated above, we specify the fundamental value in two ways: a random walk and a Gordon procedure. Following we explain the latter. Defining d_t as the dividend flow, g the average growth rate of dividends, and r the average required return, the fundamental value of asset price can be defined as:

$$P_t^f = u_t = d_t \frac{(1+g)}{(r-g)}$$

Following Chiarella et al. (2012), we assume that r is equal to the sum of the average dividend yield $\tilde{y}$ and the average rate of capital gain $\tilde{x}$. The Gordon growth model implies that x is equal to g, so to obtain:

$$u_t = d_t \frac{(1+g)}{\tilde{y}} \tag{16}$$

From Eq. (16), the fundamental value is now equal to the current dividend flow multiplied by a constant multiplier.

In terms of the state space model, the fundamental value is no more an unobservable state variable but an explanatory variable that enters directly into the observation equation of the model:

$$P_t = u_t + \begin{pmatrix} \gamma & 1-\gamma \end{pmatrix} \begin{pmatrix} B_t^f \\ B_t^c \end{pmatrix}$$

with the following transition equation

$$\begin{pmatrix} B_t^f \\ B_t^c \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} B_t^f \\ B_t^c \end{pmatrix} + \begin{pmatrix} \varphi_t \\ \eta_t \end{pmatrix}$$

As before, in a compact form, we define

$$P_t = H' Z'_t$$

$$Z'_t = A' Z'_{t-1} + \delta_t \qquad \delta'_t \sim \mathrm{N}\left(0, Q'\right)$$

where P_t is the observable asset price,

$$Z'_t = \begin{pmatrix} B_t^f \\ B_t^c \end{pmatrix}$$

is the new state vector,

$$H' = \begin{pmatrix} \gamma & 1-\gamma \end{pmatrix}$$

is the new measurement matrix,

$$A' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is the new transition matrix and δ'_t is the vector containing the state disturbance of unobserved components, normally distributed with mean zero and variances collected in the diagonal matrix Q'. Still, the dynamics of the system is consistently given by the transition equation. However, eigenvalues analysis for cyclical conditions (cyclical analysis in the transition matrix) does not change.

Once the model is in a state space form, the cyclical parameters $(a_{11}, a_{12}, a_{21}, a_{22})$ with the the proportion of fundamentalist and chartist traders $(\gamma, 1-\gamma)$ are estimated by maximum likelihood estimation via the Kalman Filter. Precisely, the parameters of the model are estimated by maximum likelihood using the prediction error decomposition approach where the one-step prediction and updating equations are calculated via the state space form using the Kalman filter (Harvey, 1989). This filter is an algorithm that uses a series of an observed variable to produces estimates of unknown variables that generate the dynamics of the observed variable. After the estimation process, we can see from the data, if the cyclical fluctuations are an endogenous outcome that we find as the result of the interaction of the two unobserved components. Moreover, after estimating cyclical parameters with the proportions of agents, from (12), we can obtain the values of reaction parameters both for fundamentalists and chartists. Depending on the signs of the coefficients, we can set the base price. Finally, the iterative Kalman filter algorithm (also called one-sided filter) is used to extract the unobserved chartist and the fundamentalist states.

3 Estimation Results

This section sets out the results of our examination. Firstly, we present the obtained results when fundamental value is considered as an unobserved component state in the state space model. Secondly, we present the results from the model where fundamental value is defined by the Gordon growth model. Finally, in the conclusions, we make a comparison between the two specifications.

3.1 Results with a Fundamental value as random walk

Table 1 shows the obtained results for time series at quarterly frequency. As it turns out, the signs of $a_{11}, a_{12}, a_{21}, a_{22}$ respect conditions in (15) for oscillatory phenomena. The parameters are inside the allowed range size $(a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0)$ and estimates are statistically significant at the one percent level. In particular, we have damped fluctuations $(a_{11}a_{22} - a_{12}a_{21} < 1)$ with the modulus near to one $(\rho = 0.9971)$, likely to generate almost constant amplitude cycles. Looking at the percentage of the two different types of agents, we notice that chartists

$(1 - \gamma)$ are the minority in comparison with fundamentalists (γ). To be more specific, 86% of the agents are estimated to be fundamentalists while the 14% are chartists; it is worth noting that the percentage of fundamentalists and chartists agents is statistically significant at the one percent level. Once we obtain these estimates, from Eq. (12) it is possible to recover the values of λ and β. The mean-reverting parameter for fundamentalists is 0.93 with $\lambda = \tilde{\lambda}$, which corresponds to a base price $P^b_{t-1} = P_{t-1}$, while the extrapolative parameter of chartists is -0.96 with $\beta = \tilde{\beta}$.

The estimated model is evaluated with diagnostic checks on residuals. Residuals should satisfy the following three properties, which, in decreasing order of importance (Commandeur and Koopman, 2007), are independence, homoscedasticity and normality. The assumption of independence of the residuals is examined with the Ljung-Box Q-Test, which tests the null hypothesis of jointly zero autocorrelation against the alternative of at least one nonzero autocorrelation. The assumption of residual homoscedasticity is checked with the Engle ARCH test to evaluate the null hypothesis of no autoregressive conditional heteroscedasticity effects. Finally, the Jarque-Bera test is used to check the null hypothesis that residuals come from a normally distributed population. In state space models, these tests are applied to what are known as the standardised prediction errors, which are the ratio between the one-step ahead prediction errors, obtained from the Kalman filter procedure, and its variance (Commandeur and Koopman, 2007).

Table 1: Estimation Results (S&P500 Quarterly Data) [Random Walk Fundamental Value]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
$Values$	-0.80966***	-0.55324***	0.83363***	-0.66194***
	(0.00012)	(0.00015)	(0.00026)	(0.00049)
	Percentage and Reaction Coefficients			
	γ	$1-\gamma$	$\lambda = \tilde{\lambda}$	$\beta = \tilde{\beta}$
$Values$	0.86***	0.14***	0.93	-0.96
	(0.02746)	(0.02746)		
	State Disturbance			
	σ_φ	σ_η	σ_ε	
$Values$	0.00123**	0.00038***	0.07893***	
	(0.00057)	(0.00014)	(0.00543)	
	Cyclical Conditions			
$[a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0]$	$Fulfilled$			
$[a_{11}a_{22} - a_{12}a_{21} < 1]$	$Damped$			
	Diagnostic Checks			
	Pvalue	*CValue*	*Stat*	
Autocorrelation Test	0.8122	63.6907	32.0023	
Heteroscedasticity Test	0.8618	63.6907	30.4684	
Normality Test	0.0020	12.3690	23.7002	
	Info Model			
Base Price	$P_{t-1}^b = P_{t-1}$			
Eigenvalues	$-0.7358 \pm 0.6751\mathrm{i}$			
Modulus	0.9971			
Log-likelihood	127.351			
Akaike Info Criterion	-236.703			
Bayesian Info Criterion	-211.32			

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Cv and *Stat* are respectively the critical value and the test statistics.

Table 1 presents the obtained results. As we can notice, there is no evidence of serial correlation and heteroscedasticity. In particular, looking at the p-values of the Ljung Box-Q statistic and comparing the critical value with the test statistics, there is no evidence to reject the null hypothesis that the residuals are not autocorrelated. This is also confirmed by the sample autocorrelation function showed in Fig. (1). At the same time, there is no sign of heteroscedasticity. Conversely, the normality test statistic is lower than the critical value, so results indicate rejection of the null hypothesis of normality.

Overall, the obtained results provide empirical support for the existence of endogenous financial cycles in equity prices as a consequence of the different expectation rules defined in our model. The qualitative differences between heuristic unobserved states can be detected by the filtered estimate of the state variables obtained via the iterative Kalman filter (Figure 2).

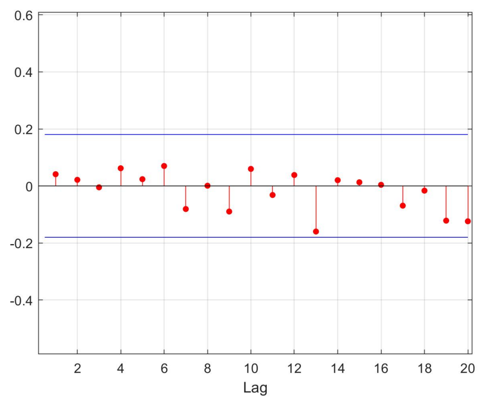

Figure 1: Sample Autocorrelation Function (Quarterly Data)

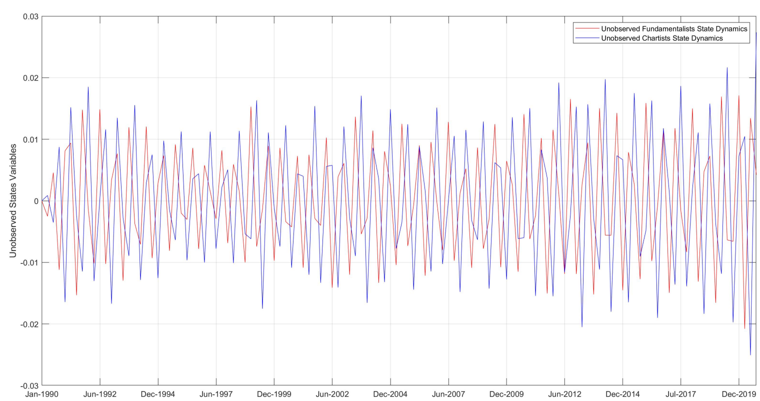

Figure 2: Filtered Unobserved Belief Dynamics (Quarterly Data)

The estimates of our monthly model are displayed in Table 2. Starting with the maximum likelihood estimates, we notice that the cyclical parameters are highly significant, with estimated coefficients significant at the one percent statistical level. Moreover, the size for $a_{11}, a_{12}, a_{21}, a_{22}$ respect conditions for damped fluctuations $(a_{11}a_{22} - a_{12}a_{21} < 1)$. As before, these results confirm the presence of endogenous fluctuation generated by the presence of different beliefs of the two group of agents.

With monthly data, 67% of agents are estimated to be fundamentalists and the remaining 33% are chartists. Again, the base price is $P^b_{t-1} = P_{t-1}$ with a value for $\lambda = \tilde{\lambda}$ equal to 0.57 and -0.52 for $\beta = \tilde{\beta}$.

The residual diagnostic can also be found in Table 2. The Ljung-Box Q-Test and sample auto-correlation function (Appendix A) suggest that the residuals do not show significant evidence of auto-correlation. Moreover, the reported critical values and test statistics in Table 2 indicate that residuals do not seem to suffer from heteroscedasticity. As before, the Jarque-Bera test rejects the null hypothesis of normality. In Appendix B, Fig. 11 plots the evolution of fundamentalists and chartists states dynamics over time.

The results for time series at daily frequency are presented in Table 3. Although the conditions for a cycling mechanism are formally satisfied with damped fluctuations, compared with the previous cases, these results are not statistically significant. The chartists correspond to 49%, while the fundamentalists are estimated to be 51%. Equal to the cyclical parameters, these percentages are not statistically significant. Again we can obtain the extent of price overshooting and mean-reverting parameters, which are 0.19 and -0.51 respectively. In contrast to the previous results, with daily data, diagnostic tests for the normality, homoscedasticity and independence of the residuals suggest that assumptions are not met. For this reason, we justify the use of quarterly and monthly data respect to daily data in terms of better diagnostic tests on the residuals. The dynamics of the unobserved strategies are again illustrated in Fig. 12.

Table 2: Estimation Results (S&P500 Monthly Data) [Random Walk Fundamental Value]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
$Values$	-0.38842***	-1.42107***	0.35466***	-1.27017***
	(0.00096)	(0.00806)	(0.00201)	(0.00184)
	Percentage and Reaction Coefficients			
	γ	$1-\gamma$	$\lambda = \tilde{\lambda}$	$\beta = \tilde{\beta}$
$Values$	0.67***	0.33***	0.57	-0.52
	(0.00710)	(0.00710)		
	State Disturbance			
	σ_φ	σ_η	σ_ε	
$Values$	0.00001***	0.00013	0.04262***	
	(0.00001)	(0.00005)	(0.00197)	
	Cyclical Conditions			
$[a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0]$	$Fulfilled$			
$[a_{11}a_{22} - a_{12}a_{21} < 1]$	$Damped$			
	Diagnostic Checks			
	Pvalue	*CValue*	*Stat*	
Autocorrelation Test	0.9119	63.6907	28.5429	
Heteroscedasticity Test	0.0296	63.6907	58.4887	
Normality Test	0.001	11.1044	77.6514	
	Info Model			
Base Price	$P_{t-1}^b = P_{t-1}$			
Eigenvalues	$-0.8293 \pm 0.0.556\mathrm{i}$			
Modulus	0.9974			
Log-likelihood	633.645			
Akaike Info Criterion	-1249.29			
Bayesian Info Criterion	-1214.02			

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Cv and *Stat* are respectively the critical value and the test statistics.

Table 3: Estimation Results (S&P500 Daily Data) [Random Walk Fundamental Value]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
$Values$	-0.09890 (0.88910)	-1.42457 (1.37512)	0.25270 (0.76142)	-1.12552 (1.24237)
	Percentage and Reaction Coefficients			
	γ	$1-\gamma$	$\lambda = \tilde{\lambda}$	$\beta = \tilde{\beta}$
$Values$	0.51 (0.57294)	0.49 (0.57294)	0.19	-0.51
	State Disturbance			
	σ_{φ}	σ_{η}	σ_{ε}	
$Values$	0.00267 (0.00806)	0.00022 (0.00402)	0.01083*** (0.00113)	
	Cyclical Conditions			
$[a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0]$	$Fulfilled$			
$[a_{11}a_{22} - a_{12}a_{21} < 1]$	$Damped$			
	Diagnostic Checks			
	Pvalue	*CValue*	*Stat*	
Autocorrelation Test	0.001	63.6907	151.06	
Heteroscedasticity Test	0.001	63.6907	2394.1	
Normality Test	0.001	9.3417	3789.7	
	Info Model			
Base Price	$P_{t-1}^b = P_{t-1}$			
Eigenvalues	$-0.6122 \pm 0.0.3106\text{i}$			
Modulus	0.4713			
Log-likelihood	23799.2			
Akaike Info Criterion	-47580.4			
Bayesian Info Criterion	-47517.7			

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Cv and *Stat* are respectively the critical value and the test statistics.

It is worth noticing some important facts. First, with an increase in the frequency time, the percentage of speculators $(1-\gamma)$ tends to increase. $(1-\gamma)$ is equal to 0.14 with quarterly data, 0.33 at monthly frequency, and 0.49 with daily data. These results line with the HABM literature, which presupposes an increasing presence of speculative position in the short run. However, these parameters, together with the cyclical ones, become not statistically significant on daily data. That means that the specific heuristic beliefs formalized in our model are not able to explain the dynamics of asset price on a daily temporal horizon. This result can be explained by the fact that shorter horizon extrapolative expectations are more driven by the difference of observed asset price without the influence of the fundamental value. Second, the reaction parameters tend to decrease with the contemporaneous increase of frequency time. Third, in all the three cases considered, with random walk model as unobserved fundamental value, the price base is the observed asset price at time before and not the fundamental value. Finally, our results support endogenous cycles for quarterly and monthly data, with the modulus of complex eigenvalues near to generate persistent oscillations. Nevertheless, the effectiveness of unobserved state beliefs on the observed asset price is minimal (on this point, see Appendix C). For this reason, we perform the estimation analysis with the fundamental value obtained from the Gordon growth model.

3.2 Results with a Fundamental value from the Gordon growth model

In Tables 4 and 5 we report the estimations for our model when the fundamental value is obtained from the Gordon growth model. This specification is applied for quarterly and monthly data, because daily information for the dividends are missing and they could be misleading.

For quarterly data (Table 4) cyclical conditions are not sensitive after relaxing the random walk hypothesis. The cyclical conditions are still respected, with the modulus of complex eigenvalues equal to 0.9489. Only a_{12} is not significant, while a_{11}, a_{21}, a_{22} remain statistically significant at one percent level. In this case, we find a similar proportion for the chartists and fundamentalists. These percentages, both at 1% statistical level, are estimated to be 0.58 for the fundamentalists while 0.42 for the chartists. From Eq. (12) the mean-reverting parameter for fundamentalists is -0.16 with $\lambda = \tilde{\lambda} - 1$, which corresponds to a base price $P^b_{t-1} = P^f_{t-1}$ and a value of $\tilde{\lambda} = 0.84$. The reaction parameters of chartists is -2 with $\beta = \tilde{\beta} - 1$ which corresponds to a value of $\tilde{\beta}$ equal to -1. Table 4 also reports diagnostic tests. As in the case of random walk, the diagnostic tests performed on the standardized prediction errors do not reject the null hypothesis of no auto-correlation and homoscedasticity. Conversely, the test fails to reject the null hypothesis of normality.

In Figures 3 and 4, it is possible to observe the unobserved state dynamics and how fundamentalists and chartists react to the difference between the observed asset price and the fundamental value as formalized in section 2.

Table 4: Estimation Results (S&P500 Quarterly Data) [Gordon Fundamental Value]

	Cyclical Parameters					
	a_{11}	a_{12}	a_{21}	a_{22}		
$Values$	1.10335*** (0.01977)	0.06899*** (0.00959)	-0.26040 (0.18528)	0.84369*** (0.05472)		
	Percentage and Reaction Coefficients					
	γ	$1-\gamma$	$\lambda = \tilde{\lambda} - 1$	$\beta = \tilde{\beta} - 1$	$\tilde{\lambda}$	$\tilde{\beta}$
$Values$	0.58*** (0.09678)	0.42*** (0.04869)	-0.16	-2	0.84	-1
	State Disturbance					
	σ_φ	σ_η				
$Values$	0.12578*** (0.03467)	0.08501 (0.09650)				
	Cyclical Conditions					
$[a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0]$	$Fulfilled$					
$[a_{11}a_{22} - a_{12}a_{21} < 1]$	$Damped$					
	Diagnostic Checks					
	Pvalue	*CValue*	*Stat*			
Autocorrelation Test	0.6678	63.6907	35.6177			
Heteroscedasticity Test	0.6843	63.6907	35.2376			
Normality Test	0.001	12.3690	68.2362			
	Info Model					
Base Price	$P_{t-1}^b = P_{t-1}^f$					
Eigenvalues	-0.9735 ± 0.0333i					
Modulus	0.9489					
Log-likelihood	129.579					
Akaike Info Criterion	-241.157					
Bayesian Info Criterion	-215.847					

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Cv and *Stat* are respectively the critical value and the test statistics.

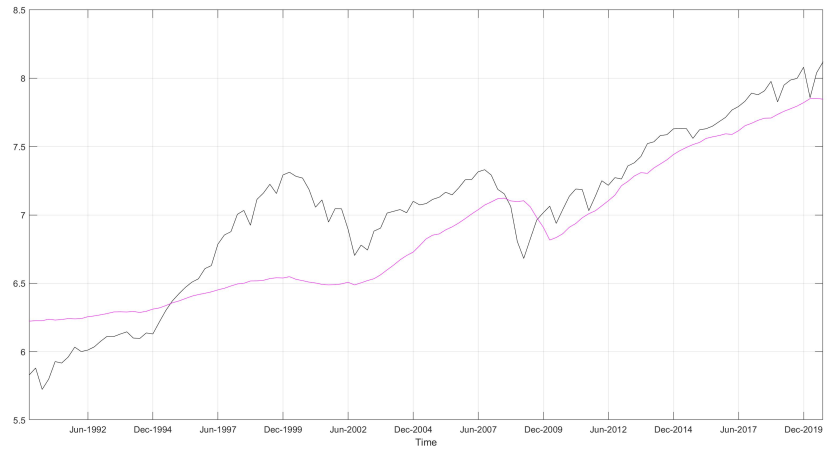

Figure 3: Stock price (black) and Fundamental Value (magenta) (Quarterly Data)

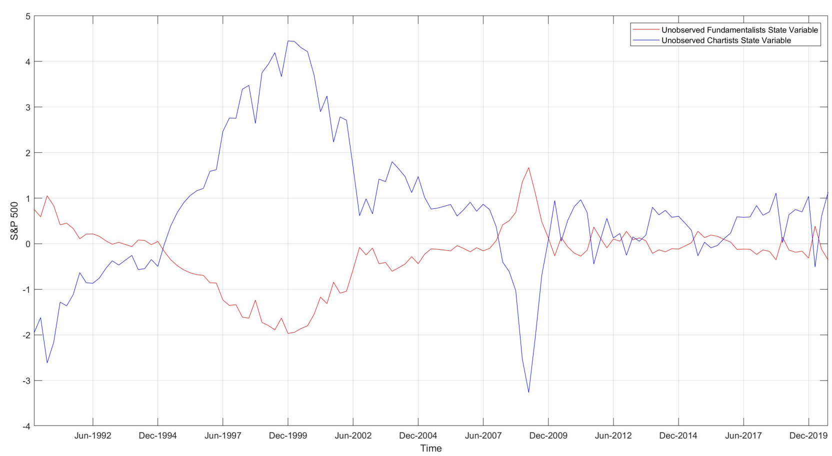

Figure 4: Filtered Unobserved Belief Dynamics (Quarterly Data)

Results for monthly data are shown in Table 5. Comparing these results with the monthly results, we notice important similarities and differences. With respect to the cyclical conditions, Eq. (15) is still validated at 1% statistical level for a_{11}, a_{21}, a_{22} and at 5% statistical level for a_{12}. In particular, we have damped fluctuations with modulus equal to 0.9795. As before, the base price remains the fundamental value at time $t-1$ such that $P^b_{t-1} = P^f_{t-1}$, but now chartists $(1-\gamma = 88\%)$ are in the majority with respect to the fundamentalists ($\gamma = 12\%$). This effect is compensated by the intensity of the reaction parameters. We observe a value for $\lambda = \tilde{\lambda} - 1$ equal to -0.42 with $\tilde{\lambda} = 0.58$ and -1.03 for $\beta = \tilde{\beta}$ such that $\tilde{\beta} = -0.03$. In this sense, with quarterly data, the speculative position is primarily taken from beliefs that are not shared by the majority of the market with a high reaction coefficient ($\beta = -1$). The opposite for monthly data, where the speculative position is primarily taken from beliefs that are shared by the majority of the market with a lower reaction coefficient ($\beta = -0.03$). Figure 6 shows the interaction between the filtered unobserved beliefs at monthly frequency and how the dynamics changes with respect the price that have been observed in the recent past. To summarize, we can conclude that estimation results provide empirical support for endogenous cycles at quarterly and monthly frequency due to the presence of heterogeneous agents. Diagnostic checks confirm the results obtained in the previous cases.

Table 5: Estimation Results (S&P500 Monthly Data) [Gordon Fundamental Value]

	Cyclical Parameters					
	a_{11}	a_{12}	a_{21}	a_{22}		
$Values$	1.06392*** (0.00063)	0.36574** (0.15674)	-0.01578*** (0.00570)	0.91519*** (0.00029)		
	Percentage and Reaction Coefficients					
	γ	$1-\gamma$	$\lambda = \tilde{\lambda} - 1$	$\beta = \tilde{\beta} - 1$	$\tilde{\lambda}$	$\tilde{\beta}$
$Values$	0.12* (0.07222)	0.88*** (0.06096)	-0.42	-1.03	0.58	-0.03
	State Disturbance					
	σ_{φ}	σ_{η}				
$Values$	0.26120 (0.20189)	0.03144 (0.05839)				
	Cyclical Conditions					
$[a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} < 0]$	$Fulfilled$					
$[a_{11}a_{22} - a_{12}a_{21} < 1]$	$Damped$					
	Diagnostic Checks					
	Pvalue	*CValue*	*Stat*			
Autocorrelation Test	0.9426	63.6907	26.9696			
Heteroscedasticity Test	0.0128	63.6907	62.5726			
Normality Test	0.001	11.1012	3881			
	Info Model					
Base Price	$P_{t-1}^b = P_{t-1}^f$					
Eigenvalues	-0.9896 ± 0.0155i					
Modulus	0.9795					
Log-likelihood	638.041					
Akaike Info Criterion	-1258.08					
Bayesian Info Criterion	-1222.81					

Notes: Standard errors in parentheses.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Cv and *Stat* are respectively the critical value and the test statistics.

Figure 5: Stock price (black) and Fundamental Value (magenta) (Monthly Data)

Figure 6: Filtered Unobserved Belief Dynamics (Monthly Data)

1

4 Some Conclusions and Open Questions

The behavioral approach in economics is now mature. Its theoretical applications date back to the end of the last century, undermining two important theoretical pillars: market efficiency based on rational behavior, and homogeneous investors. Nowadays, it is recognized that economic agents do not form their expectations as rational agents having superior cognitive capacities causing no bubbles and crashes. These not observable self-fulfilling beliefs produce waves of optimism and pessimism, making the economy more systemically fragile and possibly giving rise to cycles. In this sense, a non-stationary economy must experience at least some fleeting moments of disequilibrium so that observed prices depend on "the state of the market". At the same time, the homogeneity of investors is substituted by the presence of interacting heterogeneous agents. These concepts are the key elements of the HAM approach applied to financial markets.

In this paper, using a simple stylized HAM, we employ a state space model in order to analyze the presence of cycles within S&P500, using as fundamental value either as a random walk or with Gordon's approach. Moreover, we analyze the model for different time span, specifically daily, monthly and quarterly data.

With fundamental value as a random walk or defined by the Gordon growth model, the percentage of chartists tends to increase in correspondence of an increase of frequency time. At the same time, the intensity of both the reaction parameters decreases passing from quarterly to monthly data. With the second specification, the percentage of speculators in the market is increasing with respect to the first case with a higher percentage in the last case studied. Secondly, the base price changes in the two cases considered. With the second approach, the observed fundamental value becomes the variables that the agents consider in formalizing their heuristic expectations. Finally, and maybe the most important thing, the results with exogenous fundamental value are consistent with those obtained from the random walk as regards our main question, the existence of endogenous cycles in asset prices. However, Akaike and Bayesian info criterion suggest the superior of the second specification compared to the first one, both with quarterly and monthly data.

We think the use of the state space models is promising. From a theoretical point of view, this tool allows us to analyze the effect of unobservable components (heuristic decisions) on the dynamics of the observable components. From the empirical point of view, through the filtering procedure, it allows us to estimate the unobservable components and their associated parameters. More specifically, with this instrument, we can test not only if the interaction of the two main types of expectations governs the dynamics of financial markets but also check if there is empirical evidence of endogenous financial cycles. Moreover, as stated by Thomas LuX: moment-based methods could be applied for parameter estimation, but they would not provide an avenue for filtering information on unobserved states. So with the state space model we can directly extract the behavior of the agents.

In the light if these considerations, as future research lines we would like to replicate our analysis for other price indices, such as the Nikkei, the Dax or the FTSE. This extension would give the possibility to make a comparison between different financial markets. On the other hand, we believe it may be of value to include in the analysis other types of strategies adopted, such as strong or weak trend followers.

Finally, we would like to remind the Cars Hommes' words: *'The search for a (large) computational agent-based*

HAM capturing the stylized facts as closely as possible deserves high priority. But at the same time one would like to find the simplest behavioral HAM (e.g. in terms of number of parameters and variables), with a plausible behavioral story at the micro level, that still captures the most important stylized facts observed at the aggregate level. The simplest HAM can then be used to estimate behavioral heterogeneity in laboratory experimental and/or empirical time series data. Simple and parsimonious HAMs can thus help to discipline the wilderness of agent-based modelling'.

We believe our paper goes in this direction.

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Appendix A

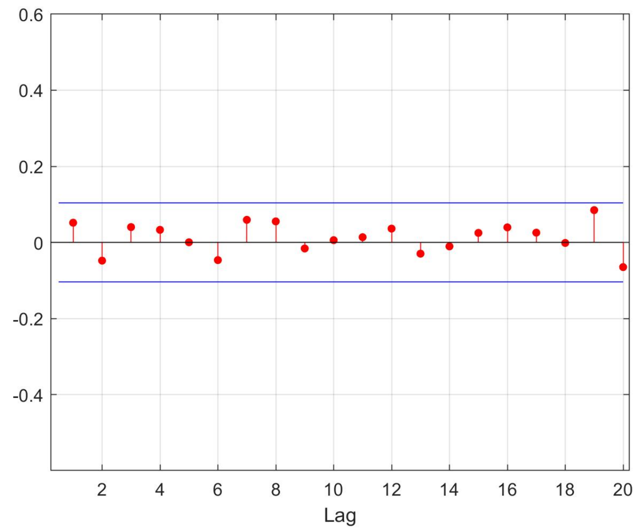

Figure 7: Sample Autocorrelation Function (Monthly Data) - Random Walk Fundamental Value

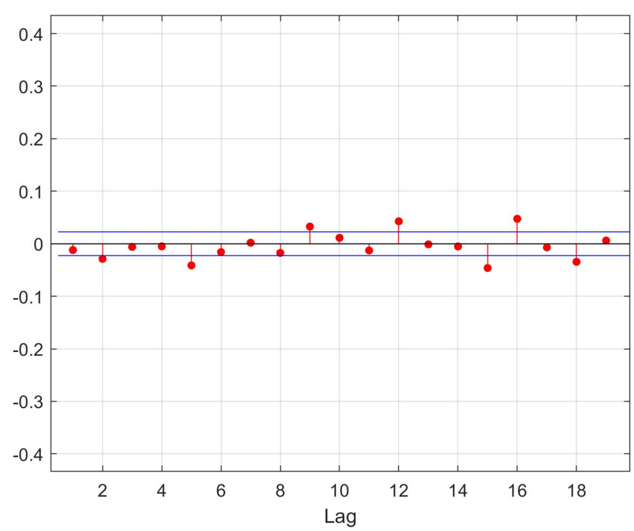

Figure 8: Sample Autocorrelation Function (Daily Data) - Random Walk Fundamental Value

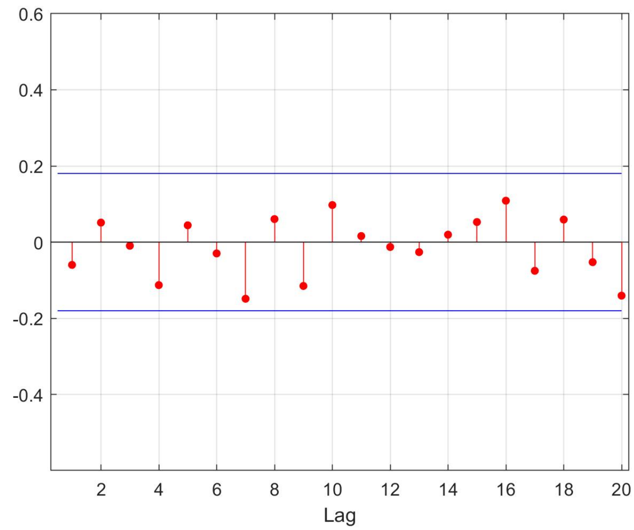

Figure 9: Sample Autocorrelation Function (Quarterly Data) - Gordon Fundamental Value

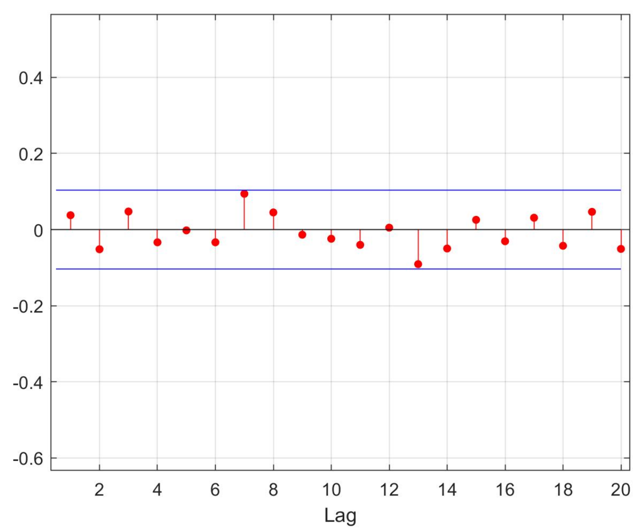

Figure 10: Sample Autocorrelation Function (Monthly Data) - Gordon Fundamental Value

Appendix B

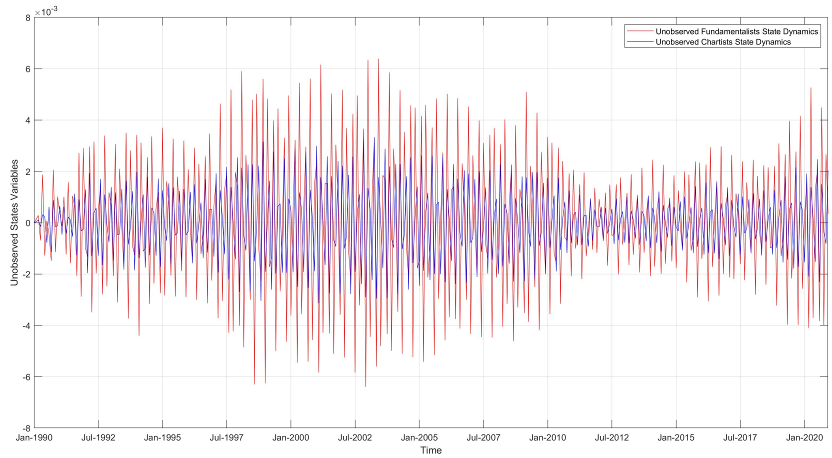

Figure 11: Filtered Unobserved Belief Dynamics (Monthly Data)

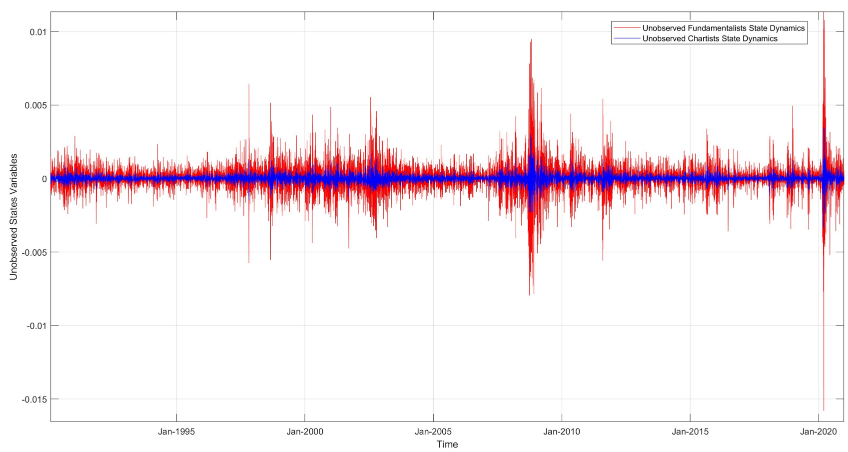

Figure 12: Filtered Unobserved Belief Dynamics (Daily Data)

Appendix C

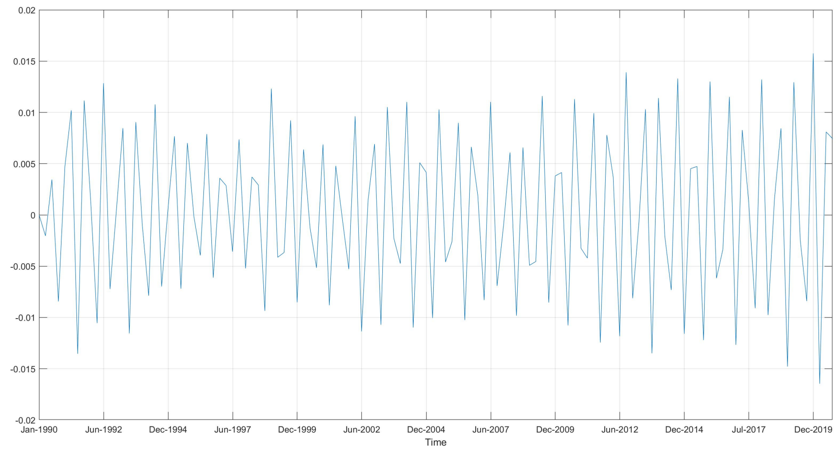

Figure 13: Difference Between Observed Asset Price and Fundamental Value (Quarterly Data)

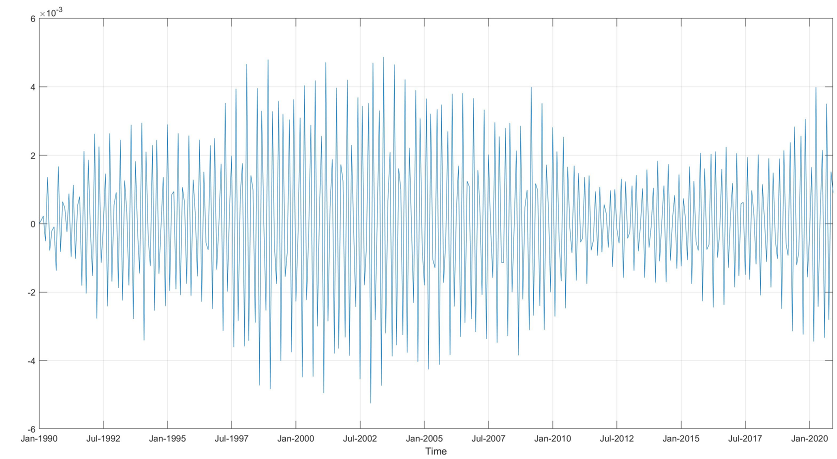

Figure 14: Difference Between Observed Asset Price and Fundamental Value (Monthly Data)

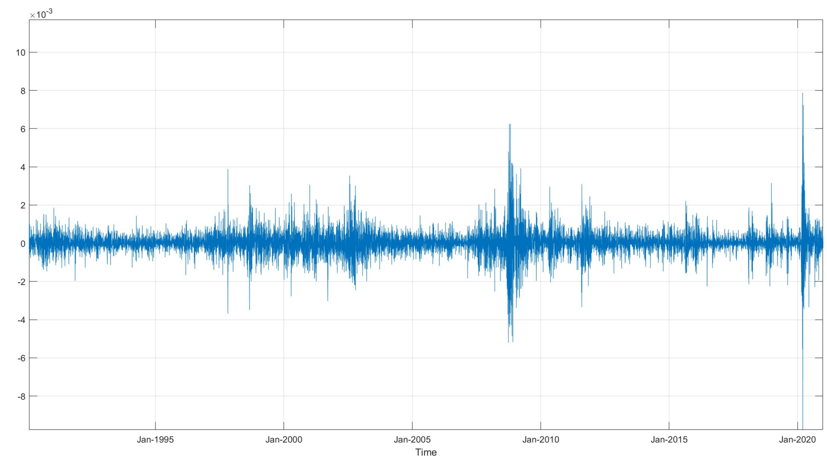

Figure 15: Difference Between Observed Asset Price and Fundamental Value (Daily Data)