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Abstract

Our main goal in this paper is to provide conditions under which the solution sets to a maximal problem and to a generalized variational inequality problem coincide and are nonempty. We use the obtained results to get a proof of existence of equilibria in an exchange economy model. Finally we compare the results obtained in our analysis with those which are available in the variational inequality and economic literature.

Keywords: General equilibrium, Variational inequality problem, Maximal problem, Preference set-valued function.
JEL Classification: C61, D11, D51.

1 Introduction

In the present work, we provide conditions under which the solution sets to a maximal problem and to a generalized variational inequality problem coincide and are nonempty. We then use the above results to get a proof of existence of equilibria in an exchange economy model.

In choice theory, individual actors are assumed to have preferences on a set of alternatives, i.e., a binary relation on a choice set. Individuals choose a subset of the choice set using such preferences[1]. The preference relation is often assumed to be transitive and complete - where completeness means the individual is able to compare any pair of alternatives. In that case, the individual will look for an element in the choice set which is preferred to any other element in the set, i.e., a maximum element.

It is indeed analytically convenient to describe preferences by a real valued function defined on the choice set. Transitivity and completeness are necessary, but not sufficient conditions to accomplish such task.[2] Debreu [12] showed that further assumptions on the preference relation (and the choice set) are required.

At the other end of the spectrum, many authors have questioned the realism of both completeness and transitivity. Indeed, completeness fails to acknowledge the possibility of being unable to choose between two alternatives, basically for lack of information. To see why transitivity may be fail in a very simple manner, observe what follows. Transitivity of preferences implies transitivity of the associated indifference relation. On the other, making a long chain of almost identical and therefore indifferent elements may clearly violate transitivity: "I am indifferent between a cup of coffee with no sugar and a cup of coffee with one grain of sugar and a cup of coffee with two grains of sugar and so on; on the other hand, I am not indifferent between a cup of coffee with no sugar and a cup with 10 spoons of sugar"[3]. In the case of preferences without completeness and transitivity, the individual will look for an element in the choice set which is not strictly preferred by any other element in the set, i.e., a maximal element.

In the present work, the way we use to find maximal elements is to follow a variational inequality approach. The theory of variational inequalities was introduced in the seventies by Stampacchia [30], inspired by work of Fichera [15], as an innovative and effective method to solve equilibrium problems arising in mathematical physics. Starting from 1985, this theory was applied to find solutions to optimization problems and system of equations. Different problems arising from the economic, financial and engineering fields (see e.g. [6],[7],[11],[13],[29] and references therein) can be modeled by means of variational and quasi-variational inequalities and can be studied by using this theory, providing existence of solutions, well posedness and iterative schemes for computational purposes.

At the best of our knowledge, our results about the maximal problem is a strict generalization of any corresponding result in the Variational Inequality literature applied to Euclidean space (see [4] and [27] and references there). The result about existence of equilibria in exchange economies is strictly more general than any work in the Variational Inequality literature and still less general than the results obtained in the economic literature. Our future research aims to fill this last gap.

The paper is organized as follows. In Section 2, we present the maximal problem, the Variational Inequality problem and our main results. In Section 3, we compare the above results with the ones available in the Variational Inequality literature. In Section 4, we apply the result obtained above to show existence of equilibria in a general equilibrium

[1] Formal definitions and statements of what said below are presented in Section 2.

[2] See, for example, [21], p.46, for a standard example of a preference relation which is transitive, complete (and satisfies also some other "reasonable" conditions) which cannot be represented by a utility function.

[3] See also [24]

model of an exchange economy. In Section 5, we make the corresponding literature comparison. The Appendix contains results used in the analysis of the above sections.

2 Existence of a Maximal element and an associated Variational Inequality problem

We start our analysis presenting first some terminology about preference relations.

Definition 1 *A binary relation on the set X is a subset R of the Cartesian product $X \times X$. If X is a choice set, we call R a preference relation. For any $x, y \in X$, we write $(x, y) \in R$ as $x \succeq y$, which we read as "x is at least as good as y" or "x is weakly preferred to y".*

Definition 2 *The strict preference relation, denoted by $\succ$, is defined as follows*
$x \succ y \Leftrightarrow x \succeq y$ but not $y \succeq x$.
and $x \succ y$ is read as "x is preferred to y".

Remark 3 *The definition of strict preference we presented above is the one most followed in the literature - see, for example, [21], page 6. Some authors give the following definition $x \succ y \Leftrightarrow$ not $y \succeq x$ - see, for example, [23], page 9.*

Definition 4 *The indifference relation, denoted by $\sim$, is defined as follows*
$x \sim y \Leftrightarrow x \succeq y$ and $y \succeq x$.
and $x \sim y$ is read as "x is indifferent to y".

Definition 5 *The preference relation $\succeq$ on a set X is*
complete if for any $x, y \in X$, either $x \succeq y$ or $y \succeq x$;
transitive if for any $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$;
continuous if X is a topological space and for any $x \in X$, both $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$ are closed set in X.

Definition 6 *$x, y \in X$ are not comparable if*

$$\text{not } x \succeq y \qquad \wedge \qquad \text{not } \ y \succeq x.$$

Definition 7 *Given a preference relation $\succeq$ on a set X, we say that*
x is maximum (for $\succeq$ on X) if $x \in X$ and for any $y \in X$, $x \succeq y$;
x is maximal (for $\succeq$ on X) if $x \in X$ and for any $y \in X$, not $(y \succ x)$.

Remark 8 *x is a maximal if $x \in X$ and for any $y \in X$, either $x \succeq y$ or x, y are not comparable, as verified below. Given $x, y \in X$, then the following cases are possible.*

		$y \succeq x$		not $y \succeq x$
$x \succeq y$		1		2
not $x \succeq y$		3		4

1. $x \succeq y$ and $y \succeq x$, i.e., $x \sim y$;
2. $x \succeq y$ and not $y \succeq x$, i.e., $x \succ y$;
3. not $x \succeq y$ and $y \succeq x$, i.e., $y \succ x$;
4. not $x \succeq y$ and not $y \succeq x$, i.e., x and y are not comparable.
The above observation suggests that if a preference relation is not complete, then a reasonable choice for an individual would be a maximal element.

Definition 9 *A function $u : X \to \mathbb{R}$ is an utility function representing the preference relation $\succeq$ if*

$$\forall x, y \in X, \ x \succeq y \Leftrightarrow u(x) \geq u(y) .$$

Proposition 10 *(See [12], page 56) If X is a connected subset of $\mathbb{R}^n$, and the preference relation $\succeq$ is complete, transitive and continuous, then there exists a continuous utility function representing the preference relation $\succeq$.*

Definition 11 *The strictly preference set-valued map $P : X \rightrightarrows X$, is defined as follows.*

$$P(x) = \{z \in X : \ \ z \succ x\} .$$

The set $P(x)$ of all vectors z that are strictly preferred to x is called strict upper contour set.

Given the above definitions, we can now present our choice problem analysis. We are interested in preference relations which are neither complete nor transitive. We consider an individual described by a choice set $X \subseteq \mathbb{R}^C$, with $C \in \mathbb{N}$, a constraint set $K \subseteq X$ and a set valued function $P : X \rightrightarrows X$. An usual economic interpretation of the above objects goes as follows. The individual is a consumer or household who has to choose a vector of C existing commodities or goods. X is the set of commodities the household can consume on the basis of biological and physical considerations; K is the set of commodities the household can afford on the basis of economic constraints depending on her wealth and existing prices; P describes preferences or tastes of the household.

Definition 12 *$\overline{x} \in X$ is a maximal element for P on K if $\overline{x} \in K$ and $P(\overline{x}) \cap K = \varnothing$. In that case, we say that $\overline{x}$ solves problem $M(K, P)$.*

Remark 13 *The above definition is consistent with Definition 7. Indeed,*

$$\begin{array}{l} x \in K \;\wedge\; \langle\, y \in K \Rightarrow y \notin P(x) \rangle \\ \Leftrightarrow \\ x \in K \;\wedge\; \langle\, K \cap P(x) = \varnothing \rangle \end{array}$$

is equivalent to

$$\begin{array}{l} \langle \exists y \in X \text{ such that } y \in K \wedge y \in P(x) \rangle \\ \Leftrightarrow \\ \langle \exists y \in X \text{ such that } y \in K \cap P(x) \rangle, \end{array}$$

which is obviously true.

Remark 14 *If $P(x) = \{x' \in X : \; u(x') > u(x)\}$, then $\overline{x} \in X$ defined above is a solution to the problem*

$$\max_{x \in K} u(x).$$

The main goal of this section is to provide conditions under which the solution sets to $M(K, P)$ and to a well chosen Generalized Variational Inequality problem defined below - see Definition 30 - do coincide and are not empty. That goal is achieved in Proposition 31 and Proposition 36.
In what follows, we present needed assumptions on X, K and P, some preliminary definitions and facts.

Definition 15 *For any $y \in \mathbb{R}^C$ and any $r \in \mathbb{R}_{++}$, the open $\mathbb{R}^C$ ball centered at y of radius r is denoted and defined as follows.*

$$\mathcal{B}(y, r) = \left\{z \in \mathbb{R}^C : \|y - z\| < r\right\},$$

the unit sphere of $\mathbb{R}^C$ centered at zero as

$$S(0, 1) = \left\{z \in \mathbb{R}^C : \|z\| = 1\right\}$$

Definition 16 *Given $K \subseteq \mathbb{R}^C$,*

$$\mathrm{Int}_{\mathbb{R}^C}(K) = \{x \in K : \exists r \in \mathbb{R}_{++} \text{ such that } \mathcal{B}(x, r) \subseteq K\},$$

$$K^{\perp} = \left\{z \in \mathbb{R}^C : \forall x, y \in K, \langle z, x - y\rangle = 0\right\},$$

$$\mathrm{aff}(K) = \left\{x \in \mathbb{R}^C : \exists m \in \mathbb{N}, (\lambda_i)_{i=1}^m \in \mathbb{R}^m,\ a^1, ..., a^m \in A \text{ such that } x = \sum_{i=1}^m \lambda_i a^i \text{ and } \sum_{i=1}^n \lambda_i = 1\right\},$$

$$\mathrm{conv}(K) = \left\{x \in \mathbb{R}^C : \exists m \in \mathbb{N}, (\lambda_i)_{i=1}^m \in \mathbb{R}_+^m,\ a^1, ..., a^m \in A \text{ such that } x = \sum_{i=1}^m \lambda_i a^i \text{ and } \sum_{i=1}^n \lambda_i = 1\right\}.$$

Definition 17 *Given $x, y \in \mathbb{R}^C$ such that $x \neq y$, the closed segment from x to y is denoted and defined as follows*

$$[x, y] = \left\{z \in \mathbb{R}^C : \exists \in \lambda \in [0, 1] \text{ such that } z = (1 - \lambda) x + \lambda y\right\}.$$

Similar definition applies to half open segments $[x, y)$.

Definition 18 *A set valued function $P : X \rightrightarrows X$ is lower semicontinuous at $x \in X$ if $P(x) \neq \varnothing$ and for every sequence $(x_n)_{n \in \mathbb{N}} \in X^{\infty}$ such*[4] *that $x_n \to x$, and for every $y \in P(x)$, there exists a sequence $(y_n)_{n \in \mathbb{N}} \in X^{\infty}$ such that $\forall\, n \in \mathbb{N}$, $y_n \in P(x_n)$ and $y_n \to y$. P is lower semicontinuous if it is lower semicontinuous at every $x \in X$.*

[4] Given a set S, S^{∞} denotes the set of sequences with elements in S.

Definition 19 *$P : X \rightrightarrows X$ is $\mathbb{R}^C$ open valued a $x \in X$ if $P(x)$ is $\mathbb{R}^C$ open. P is X open valued at $x \in X$, if there exists $r \in \mathbb{R}_{++}$ such that $\mathcal{B}(y,r) \cap X \subseteq P(x)$.*

Assumptions

(i.1) K is convex;

(i.2) $K^{\perp} = \{0\}$;

(i.2*) $K \neq \varnothing$;

(i.2**) $K \subseteq \mathrm{Int}_{\mathbb{R}^C}(X)$;

(i.3) K is compact;

(ii) P is lower semicontinuous;

(iii) (Openness like assumption) For any $x \in X$, for any $y \in P(x)$ and for any $z \in X \setminus \{y\}$, we have $[z,y) \cap P(x) \neq \varnothing$;

(iv) P is convex valued;

(v) (Irreflexivity) For any $x \in X$, $x \notin P(x)$;

(vi) (Global NonSatiation) P is non-empty valued;

(vii) For any $x \in X$, $\mathrm{Int}_{\mathbb{R}^C} P(x) \neq \varnothing$.

(vii*) For any $x \in X$, there exists $y \in P(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z,y) \cap P(x) \neq \varnothing$;

(viii) (Local NonSatiation) For any $x \in X$ and any $r > 0$, there exists $y \in X$ such that $y \in P(x) \cap \mathcal{B}(x,r)$.

Remark 20 *1. P is Locally NonSatiated $\Leftrightarrow$ For any $\overline{x} \in \mathbb{R}^C$, $\overline{x} \in \mathrm{Cl}_{\mathbb{R}^C}(P(\overline{x}))$, where* Cl *denotes the Closure of a set (with respect to the Euclidean topology of $\mathbb{R}^C$).*
2. Assumption (i.2) $\Rightarrow$ Assumption (i.2);*
Indeed, $K^{\perp} = \{0\} \Leftrightarrow \mathrm{aff}(K) = \mathbb{R}^C$ and if $K = \varnothing$, then $\mathrm{aff}(K) = Cl(\mathrm{aff}(\varnothing)) = \varnothing \neq \mathbb{R}^C$.
3. Assumption (viii) $\Rightarrow$ Assumption (vi).
4. P is $\mathbb{R}^C$ open and non-empty valued $\Rightarrow$(vii) $\Rightarrow$ (vii) $\Rightarrow$ (vi).*
5. P is X open valued $\Rightarrow$ Assumption (iii).
6. Assumption (viii) $\Rightarrow$ Assumption (vi).
7. $\mathrm{Int}_{\mathbb{R}^C}(K) \neq \varnothing \Rightarrow$ (i.2) (see Proposition 60 in the Appendix).
8. $\varnothing \neq K \subseteq \mathrm{Int}_{\mathbb{R}^C}(X) \nRightarrow \mathrm{Int}_{\mathbb{R}^C}(K) \neq \varnothing$;
indeed, it is enough to take $X = \mathbb{R}^2_{++}$ and $K = \left\{x \in \mathbb{R}^2_{++} : x_1 + x_2 = 1\right\}$
9. $\varnothing \neq K \subseteq \mathrm{Int}_{\mathbb{R}^C}(X) \nLeftarrow \mathrm{Int}_{\mathbb{R}^C}(K) \neq \varnothing$.
10. Assumption (ii) $\Rightarrow$ Assumption (vi).

Remark 21 *All the assumptions listed above have been used extensively in the literature apart from Assumptions (iii) and (vii*). To the best of our knowledge, [9], page 133, was the first author to introduce Assumption (iii). We introduced Assumption (vii*). That assumption is more general than "For any $x \in X$, $P(x)$ contains an $\mathbb{R}^C$ internal point", where $y \in P(x)$ is an $\mathbb{R}^C$ internal point if for any $z \in \mathbb{R}^C \setminus \{y\}$, there exists $t_z \in \mathbb{R}_{++}$ such that for any $t \in [0, t_z]$, we have $y + tz \in P(x)$.*

Proposition 22 *Define $\succeq$ on $\mathbb{R}$ as follows: for any $x, y \in \mathbb{R}$, $y \succeq x$ if $y \in [x, x+1)$.*
1. $\succeq$ is a. neither transitive, b. nor complete;
2. $P(x) := \{y \in \mathbb{R} : y \succ x\} = (x, x+1)$;
3. P satisfies Assumptions from (ii) to (viii).

Proof. Define

$$P_w : \mathbb{R} \to\!\!\to \mathbb{R}, \quad P_w(x) = [x, x+1).$$

Therefore, $y \succeq x$ if and only if $y \in P_w(x)$.
1.a. $2.5 \succeq 1.5$ and $1.5 \succeq 1.1$ but it is not the case that $2.5 \succeq 1.1$. b. Neither $1 \succeq 3$ nor $3 \succeq 1$.

2. We have to check that $y \succ x \Leftrightarrow y \in (x, x+1)$.

$$\begin{array}{ll} y \succ x \Leftrightarrow y \succeq x \text{ and not } x \succeq y & \Leftrightarrow \\ y \in P_w(x) \text{ and not } x \in P_w(y) & \Leftrightarrow \\ y \in [x, x+1) \wedge x \notin [y, y+1) & \Leftrightarrow \\ y \in [x, x+1) \wedge (x \in (-\infty, y) \ \vee \ x \in [y+1, +\infty)) & \Leftrightarrow \\ y \in [x, x+1) \wedge (x < y \ \vee \ x \geq y+1) & \Leftrightarrow \\ y \in [x, x+1) \wedge (y > x \vee \ y \leq x-1) & \Leftrightarrow \\ y \in [x, x+1) \wedge (y \in (x, +\infty) \cup (-\infty, x-1]) & \Leftrightarrow \\ y \in (x, x+1) & \end{array}$$

3. It is easy to check P satisfies any of the assumptions we listed about P. ■

Remark 23 *If we consider the maximization problem described in Remark 14 and u is continuous and (quasi-concave and Locally NonSatiated) or (semistrictly quasi-concave and Globally NonSatiated), then the set-valued function $P : X \rightrightarrows \mathbb{R}$, $P(x) = \{x' \in X : \ u(x') > u(x)\}$ does satisfied the Assumption listed above.*

Definition 24 *The normal cone to $P(x)$ at $x \in X$ is denoted and defined as follows.*

$$N^{>}(x) = \left\{h \in \mathbb{R}^C : \forall z \in P(x), \quad \langle h, z - x\rangle \leq 0\right\}.$$

Remark 25 *The normal cone to a convex set is a closed convex cone - see, for example, Proposition 2.10, page 42, in [28].*

Proposition 26 *If P satisfies Assumptions (iv),(v) and (vi), then*

$$\forall x \in X, \quad N^{>}(x) \setminus \{0\} \neq \varnothing.$$

Proof. From Assumptions (iv),(v) and (vi), $x \notin P(x)$ and P is convex and nonempty valued. We can then apply the Separation Theorem 62 in the Appendix and conclude that there exists

$$h \neq 0 \tag{1}$$

such that for any $z \in P(x)$, $\langle h, z - x\rangle \leq 0$, i.e., from Definition 24,

$$h \in N^{>}(x). \tag{2}$$

(1) and (2) are the desired results. ■

Remark 27 *We are going to use the next result in Proposition 36 below.*

Proposition 28 *If P satisfies Assumption (ii), then the set-valued map $N^{>} : X \rightrightarrows X$, $x \mapsto\mapsto N^{>}(x)$ is closed.*

Proof. Let $(x_k)_{k\in\mathbb{N}} \in X^\infty$ and$(y_k)_{k\in\mathbb{N}} \in X^\infty$ be sequences such for any $k \in \mathbb{N}$, $y_k \in N^{>}(x_k)$ and such that $\lim_{k\to+\infty} x_k = x$ and $\lim_{k\to+\infty} y_k = y$. We have to prove that $y \in N^{>}(x)$, that is, for any $z \in P(x)$, we have $\langle y, z - x\rangle \leq 0$. Since P is lower semicontinuous, then, taken $z \in P(x)$, there exists a sequence $(z_k)_{k\in\mathbb{N}}$ such that for any $k \in \mathbb{N}$, $z_k \in P(x_k)$ and $\lim_{k\to+\infty} z_k = z$. From the facts that $z_k \in P(x_k)$ and $y_k \in N^{>}(x_k)$, it follows $\langle y_k, z_k - x_k\rangle \leq 0$. Taking limits we get $\langle y, z - x\rangle \leq 0$, i.e., $y \in N^{>}(x)$, as desired. ■

Definition 29

$$G : X \rightrightarrows \mathbb{R}^C, \qquad G(x) = conv\left(N^{>}(x) \cap S(0,1)\right)$$

Definition 30 *Given a set $K \subseteq \mathbb{R}^C$ and a set-valued function $P : X \to\to X$, we define the Problem GVI(K, P) as follows.*

$$\text{Find } \overline{x} \in K \text{ such that } \exists\, h \in G(\overline{x}) \text{ such that } \forall x \in K, \quad \langle h, x - \overline{x}\rangle \geq 0 \tag{3}$$

Proposition 31

1. $\overline{x}$ *solves* $GVI(K,P)$ *and either a. Assumptions (iii), (iv), (v), (vii*), (i.2), or b. Assumptions (iii), (iv), (v), (vi), (i.2**) hold true* $\Rightarrow$ $\overline{x}$ *solves* $M(K,P)$;

2. $\overline{x}$ *solves* $GVI(K,P)$ $\Leftarrow$ $\overline{x}$ *solves* $M(K,P)$ *and Assumptions (i.1), (i.2*), (iv), (viii) hold true.*

To prove Proposition 31, we need some preliminary results.

Definition 32 *A cone* $K \subseteq \mathbb{R}^C$ *is pointed if* $K \cap (-K) = \{0\}$.

Proposition 33 *1. If Assumption (vii*) hold true, then for any* $\overline{x} \in X$*, we have that* $N^>(\overline{x})$ *is a pointed cone.*
2. If Assumption (iii) holds true and $\varnothing \neq P(\overline{x}) \cap K \subseteq \operatorname{Int}_{\mathbb{R}^C}(X)$*, then* $N^>(\overline{x})$ *is a pointed cone.*

Proof. 1.
Suppose otherwise, i.e., there exists $h \in \mathbb{R}^C \setminus \{0\}$ such that $h \in N^>(\overline{x}) \cap (-N^>(\overline{x}))$, or

$$\exists h \in \mathbb{R}^C \setminus \{0\} \text{ such that } h, -h \in N^>(\overline{x}).$$

From Assumption (vii*), we have that there exists $y \in P(\overline{x})$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z, y) \cap P(\overline{x}) \neq \varnothing$. Since $y \in P(\overline{x})$ and $h, -h \in N^>(\overline{x})$, we have that $\langle h, y - \overline{x} \rangle \leq 0$ and $\langle -h, y - \overline{x} \rangle \leq 0$, i.e.,

$$\langle h, y - \overline{x} \rangle = 0. \tag{4}$$

Since $h \neq 0$, then $y + h \neq y$ and from Assumption (vii*), identifying $x \in X, y \in P(x), z \in \mathbb{R}^C \setminus \{y\}$ with $\overline{x} \in X, y \in P(\overline{x}), y + h \in \mathbb{R}^C \setminus \{y\}$, we have that $[y + h, y) \cap P(\overline{x}) \neq \varnothing$, i.e.,

$$\exists \overline{\alpha} \in (0, 1] \text{ such that } y + \overline{\alpha} h \in P(\overline{x}). \tag{5}$$

Then,

$$0 \overset{\text{def. } N^>(\overline{x}),\ (5)}{\geq} \langle h, y + \overline{\alpha} h - \overline{x} \rangle = \langle h, \overline{y} - \overline{x} \rangle + \overline{\alpha} \|h\|^2 \overset{(4)}{=} \overline{\alpha} \|h\|^2 > 0,$$

the desired contradiction.
2. (The proof is similar to the proof of 1. above)
Suppose otherwise, i.e.,

$$\exists h \in \mathbb{R}^C \setminus \{0\} \text{ such that } h, -h \in N^>(\overline{x}).$$

Since by assumption $\varnothing \neq P(\overline{x}) \cap K \subseteq \operatorname{Int}_{\mathbb{R}^C}(X)$, we can take $z \in P(\overline{x}) \cap K \subseteq \operatorname{Int}_{\mathbb{R}^C}(X)$. Then, since $z \in P(\overline{x})$ and $h, -h \in N^>(\overline{x})$, we have that $\langle h, z - \overline{x} \rangle \leq 0$ and $\langle -h, z - \overline{x} \rangle \leq 0$, i.e.,

$$\langle h, z - \overline{x} \rangle = 0. \tag{6}$$

Since $h \neq 0$, then $z + h \neq z$ and since $z \in \operatorname{Int}_{\mathbb{R}^C}(X)$, then there exists $\alpha_h \in \mathbb{R}_{++}$ such that $z + \alpha_h h \in X \setminus \{z\}$. Then, from Assumption (iii), identifying $x \in X, y \in P(x), z \in X \setminus \{y\}$ with $\overline{x} \in X$, $z \in P(\overline{x})$, $z + \alpha_h h \in X \setminus \{z\}$, we get $[z + \alpha_h h, z) \cap P(\overline{x}) \neq \varnothing$. Then

$$\exists \overline{\alpha} \in (0, \alpha_h] \text{ such that } y + \overline{\alpha} h \in P(\overline{x}). \tag{7}$$

Then,

$$0 \overset{\text{def. } N^>(\overline{x}),\ (7)}{\geq} \langle h, y + \overline{\alpha} h - \overline{x} \rangle = \langle h, \overline{y} - \overline{x} \rangle + \overline{\alpha} \|h\|^2 \overset{(4)}{=} \overline{\alpha} \|h\|^2 > 0,$$

the desired contradiction. ■

Proposition 34 *If Assumptions (iv), (v) and (vi) hold true, then for any* $x \in X$, *there exists* $h' \neq 0$ *such that* $h' \in G(x)$.

Proof. Since Assumptions (iv), (v) and (vi) hold true, then we can use Proposition 26 and take $h \in N^>(x) \setminus \{0\}$. Then, by definition of $N^>(x)$, we have that

$$\forall y \in P(x), \quad \langle h, y - x \rangle \leq 0.$$

Since $h \neq 0$, we can define $h' = \frac{h}{\|h\|}$. Then,

$$\forall y \in P(x), \ \langle h', y - x \rangle = \frac{1}{\|h\|} \ \langle h, y - x \rangle \leq 0,$$

i.e., $h' \in N^{>}(x)$ and moreover $h' := \frac{h}{\|h\|} \in S(0,1)$. Therefore

$$h' \neq 0 \text{ and } h' \in N^{>}(x) \cap S(0,1) \subseteq \operatorname{conv}\left(N^{>}(x) \cap S(0,1)\right) := G(x).$$

■

Proposition 35 *If*
$x \in K$ and $P(x) \cap K \neq \varnothing$, and
either Assumptions (iv), (v), and (vii) hold true,*
*or Assumptions (iii), (iv), (v), (vi), (i.2**) hold true,*
then

$$0 \notin G(x).$$

Proof. Since $\varnothing \neq P(x) \cap K \subseteq K$, if (i.2**) holds true, then we have $\varnothing \neq P(x) \cap K \subseteq K \subseteq \operatorname{Int}_{\mathbb{R}^C}(X)$.
Then, under the assumptions of the present Proposition, both sets of assumptions of Proposition 33 hold true and we can conclude that $N^{>}(x)$ is a pointed cone. Then,

$$N^{>}(x) \cap \left(-N^{>}(x)\right) = \{0\}. \tag{8}$$

Now, suppose our claim is false, i.e.,

$$0 \in \ \operatorname{conv}\left(N^{>}(x) \cap S(0,1)\right).$$

Since Assumptions (iv), (v), (vi) hold true (recall that (vii*) implies (vi)), we can apply Proposition 34. Therefore, there exists $h' \neq 0$ such that $h' \in G(x) := \operatorname{conv}(N^{>}(x) \cap S(0,1))$. Then since $0, h \in G(x)$ and $h \neq 0$, then $\dim \operatorname{conv}(N^{>}(x) \cap S(0,1)) := r \geq 1$. From Caratheodory's theorem - see Theorem 2.2.4, page 55, Webster (1994) - we have that there exist

$$\lambda = (\lambda_i)_{i=1}^{r+1} \in \Delta_{r+1} := \left\{ x \in \mathbb{R}_+^{r+1} : \sum_{i=1}^{r+1} x_i = 1 \right\}$$

and

$$\{v_1, ..., v_i, ..., v_{r+1}\} \subseteq N^{>}(x) \cap S(0,1) \tag{9}$$

such that

$$\sum_{i=1}^{r+1} \lambda_i v_i = 0. \tag{10}$$

Since $\lambda \in \Delta_{r+1}$, then there exists $s \in \{1, ..., r+1\}$ such that $\lambda_s > 0$. Then, from (10), we have

$$N^{>}(x) \overset{(9)}{\ni} v_s = - \sum_{i \in \{1,...,r+1\} \setminus \{s\}} \frac{\lambda_i}{\lambda_s} v_i \tag{11}$$

Observe that since $r \geq 1$, then $\{1, ..., r+1\} \setminus \{s\} \neq \varnothing$. Recall that A is a convex cone if and only if for any $a, b \in A$ and any scalar $\lambda, \mu \geq 0, \lambda a + \mu b \in A$ - see Proposition 63 in the Appendix. Then, since for any $i \in \{1, ..., r+1\} \setminus \{r\}$, $\frac{\lambda_i}{\lambda_s} \geq 0$, we have

$$-v_s = \sum_{i \in \{1,...,r+1\} \setminus \{s\}} \frac{\lambda_i}{\lambda_s} v_i \in N^{>}(x). \tag{12}$$

Since from (9), $v_s \in S(0,1)$, we also have

$$v_s \neq 0. \tag{13}$$

(11), (12) and (13) contradict (8). ■

Proof. of Proposition 31.
Before proving statements 1.a. and 1.b, we present some preliminary observations.
1.
Since by assumption $\overline{x}$ solves GVI (K, P), then

$$\overline{x} \in K \text{ and } \exists h \in G(\overline{x}) \text{ such that } \forall x \in K, \ \langle h, x - \overline{x} \rangle \geq 0. \tag{14}$$

Then,

$$h \in G(\overline{x}) := \operatorname{conv}\left(N^{>}(\overline{x}) \cap S(0,1)\right) \subseteq \operatorname{conv}\left(N^{>}(\overline{x})\right) = N^{>}(\overline{x}), \tag{15}$$

where last equality follows from Remark 25 and the fact that we assumed (iv).
Now assume our claim is false, i.e., since from (14), $\overline{x} \in K$,

$$\overline{x} \in K \text{ and } P(\overline{x}) \cap K \neq \varnothing. \tag{16}$$

Then, from (16), and since Assumptions (iv), (v) and (vii*) hold true, or Assumptions (iii), (iv), (v), (vi) and (i.2**) hold true, we can apply Proposition 35. Therefore, $0 \notin G(\overline{x})$ and since $h \in G(\overline{x})$, we have that

$$h \neq 0. \tag{17}$$

From (15) and (17), we have that

$$h \in N^{>}(\overline{x}) \setminus \{0\}. \tag{18}$$

Since $h \in N^{>}(\overline{x})$, we have that

$$\forall w \in P(\overline{x}), \quad \langle h, w - \overline{x} \rangle \leq 0. \tag{19}$$

a.
Since by Assumption (i.2), $K^{\perp} = \{0\}$ and from (17) $h \neq 0$, then $h \notin K^{\perp}$, i.e., there exists $y \in K$ such that $\langle h, y - \overline{x} \rangle \neq 0$. From (14),

$$\exists y \in K \text{ such that } \langle h, y - \overline{x} \rangle > 0. \tag{20}$$

From (16), there exists $x \in X$ such that

$$x \in P(\overline{x}) \cap K. \tag{21}$$

Now observe that $y \neq x$. Indeed, if $y = x$, then from (20), we would have $\langle h, x - \overline{x} \rangle > 0$. Since from (21), we have $x \in P(\overline{x})$ and $h \in N^{>}(\overline{x})$, we also have $\langle h, x - \overline{x} \rangle \leq 0$, a contradiction.
Since, from (21), $x \in P(\overline{x})$ and $y \neq x$, we can use Assumption (iii), identifying $x \in X, y \in P(x), z \in X \setminus \{y\}$ there with $\overline{x} \in X, x \in P(\overline{x}), y \in X \setminus \{x\}$ here, respectively to conclude that

$$[y, x) \cap P(\overline{x}) \neq \varnothing. \tag{22}$$

For any $\lambda \in (0,1]$, define

$$x(\lambda) = (1-\lambda)x + \lambda y.$$

Then

$$\langle h, x(\lambda) - \overline{x} \rangle = \langle h, \ (1-\lambda)x + \lambda y - \overline{x} + \lambda\overline{x} - \lambda\overline{x} \rangle = \lambda \langle h, y - \overline{x} \rangle + (1-\lambda) \langle h, x - \overline{x} \rangle > 0,$$

where the strict inequality follows from (20), (14) and the fact that $\lambda \in (0,1]$. Therefore, from (19), $\forall \lambda \in (0,1]$, $x(\lambda) \notin P(\overline{x})$, i.e.,

$$[y, x) \cap P(\overline{x}) = \varnothing,$$

contradicting (22).
b.
From (16) and (14),

$$\exists x' \in P(\overline{x}) \cap K \subseteq K \text{ such that } \langle h, x - \overline{x} \rangle \geq 0 \tag{23}$$

and from (18),

$$h \in N^{>}(\overline{x}) \setminus \{0\}. \tag{24}$$

Since $h \in N^{>}(\overline{x})$ and since $x' \in P(\overline{x})$, we have $\langle h, x' - \overline{x} \rangle \leq 0$, and therefore

$$\langle h, x' - \overline{x} \rangle = 0. \tag{25}$$

From Assumption (i.2**) and from (23), we have that there exists $\varepsilon > 0$ such that $\mathcal{B}(x', \varepsilon) \subseteq X$ and therefore, since, from (24) $h \neq 0$, there exists $\alpha > 0$ such that $x' + \alpha h \in X$. Since, from (23), $x' \in P(\overline{x})$ and $\overline{x} \neq x'$, we can use Assumption (iii) to conclude that $[x' + \alpha h, x') \cap P(\overline{x}) \neq \varnothing$, i.e., there exists $\alpha^* \in (0, \alpha)$ such that $x' + \alpha^* h \in P(\overline{x})$. Then, since $h \in N^{>}(\overline{x})$, $\langle \ h, (x' + \alpha^* h) - \overline{x} \rangle \leq 0$. Finally,

$$0 \geq \langle \ h, (x' + \alpha^* h) - \overline{x} \rangle = \langle \ h, x' - \overline{x} \rangle + \alpha^* \|h\|^2 \overset{(24),(25)}{=} \alpha^* \|h\|^2 > 0,$$

the desired contradiction.
2.
By assumption

$$P(\overline{x}) \cap K = \varnothing, \tag{26}$$

and by Assumptions (i.1), (1.2*), (iv) and (viii) - which implies (vi),

$$K \text{ and } P(\overline{x}) \text{ are nonempty and convex.}$$

Then, from the Separation Theorem 62 in the Appendix - we have that

$$\exists h \in \mathbb{R}^C \setminus \{0\} \quad \text{such that} \quad \forall s \in P(\overline{x}) \text{ and } \forall t \in K, \quad \langle h, s \rangle \leq \langle h, t \rangle \text{ or } \langle h, t - s \rangle \geq 0. \tag{27}$$

From Remark 20.1, and Assumptions (viii), we have that $\overline{x} \in \mathrm{Cl}\ (P(\overline{x}))$. Therefore, there exists $(x_k)_{k \in \mathbb{N}} \in (P(\overline{x}))^{\infty}$ such that $x_k \to \overline{x}$. Then,

$$\exists h \in \mathbb{R}^C \setminus \{0\} \quad \text{such that } \forall k \in \mathbb{N},\ \forall t \in K, \qquad \langle h, t - x_k \rangle \geq 0,$$

Taking limits, we get

$$\exists h \in \mathbb{R}^C \setminus \{0\} \quad \text{such that } \forall t \in K, \quad \langle h, t - \overline{x} \rangle \geq 0. \tag{28}$$

From (27) and since $\overline{x} \in K$ - because by assumption, $\overline{x}$ solves GVI (K, P) - we have that

$$\exists h \in \mathbb{R}^C \setminus \{0\} \quad \text{such that } \forall s \in P(\overline{x}), \quad \langle h, s - \overline{x} \rangle \leq 0, \tag{29}$$

i.e.,

$$h \in N^{>}(\overline{x}). \tag{30}$$

From (28), we have that

$$\exists h' := \tfrac{h}{\|h\|} \neq 0 \text{ such that } \forall t \in K, \ \langle h', t - \overline{x} \rangle \geq 0. \tag{31}$$

Moreover, from Remark 25, $N^{>}(\overline{x})$ is a cone. Therefore, from (30), we do have

$$h' \in N^{>}(\overline{x});$$

moreover,

$$h' := \frac{h}{\|h\|} \in S(0,1).$$

Finally,

$$h' \in N^{>}(\overline{x}) \cap S(0,1) \subseteq \operatorname{conv}\left(N^{>}(\overline{x}) \cap S(0,1)\right) := G(\overline{x}). \tag{32}$$

The fact that $\overline{x} \in M(K)$ and therefore $\overline{x} \in K$, (31) and (32) are the desired result: $\overline{x}$ is a solution to Problem GVI (K, P). ■

Proposition 36 *1. If Assumptions (i.1), (i.2*), (i.3), (ii), (iv), (v) and (vi) hold true, then*

set of solutions to GVI $(K,P) \neq \varnothing$*;*

2.a. If Assumptions (i.1), (i.2), (i.3), (ii), (iii), (iv), (v), and*
((vii)*[5] *and (i.2)) or ((vi) and (i.2.**))*
hold true, then

$\varnothing \neq$ *set of solution to GVI* $(K,P) \subseteq$ *set of solutions to* $M(K,P)$*;*

b. If in addition Assumptions (viii) holds true then

set of solution to GVI $(K,P) =$ *set of solutions to* $M(K,P) \neq \varnothing$*.*

Proof. 1.
We want to use Proposition 77 in the Appendix, identifying C, F with K, G respectively.
K is nonempty, compact and convex from Assumptions (i.1), (i.2*) and (i.3).To get the desired result we are left with showing that G is nonempty valued, compact valued, convex valued, closed and upper semicontinuous on K. From Proposition 34, which uses Assumptions (iv), (v) and (vi), G is nonempty valued. The other needed properties are verified below - using Proposition 28 which uses Assumption (ii).
a. (convex valued) $\operatorname{conv}(N^{>}(x) \cap S(0,1))$ is a convex set.
b. (compact valued) $N^{>}(x)$ is closed by definition of normal cone and by continuity of inner product; $S(0,1)$ is compact. Then, $N^{>}(x) \cap S(0,1)$ is compact as well (it is a closed subset of a compact space in $\mathbb{R}^C$). Finally, the desired result follows from the fact that the convex hull of a compact set is compact - see Proposition 73 in the Appendix.
c. (upper semicontinuous) We want to use the following result: " if $\varphi : X \rightrightarrows Y$ is a set valued function which is closed graph and if $\varphi(X)$ is contained in a compact set, then φ is upper semicontinuous." - see Proposition 75 in the appendix.
Step 1. For any $x \in X$, $G(x) \subseteq \mathrm{Cl}(B(0,1))$.

[5] Recall that (vii*) $\Rightarrow$ (vi).

Observe that $N^{>}(x) \cap S(0,1) \subseteq S(0,1) \subseteq \mathrm{Cl}(B(0,1))$. Then,

$$G(x) \subseteq \mathrm{conv}(\mathrm{Cl}(B(0,1))) = \mathrm{Cl}(B(0,1)).$$

Step 2. G is closed graph.
Assume that $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R}^C)^{\infty}$ is such that $x_n \overset{n}{\to} \overline{x}$ and $\forall n \in \mathbb{N}$, $v_n \in G(x_n)$ and $v_n \overset{n}{\to} \overline{v}$. We want to show that $\overline{v} \in G(\overline{x})$.
By assumption for any $n > N$, $v_n \in G(x_n) = \mathrm{conv}\ (N^{>}(x_n) \cap S(0,1))$. Now, from Carathèodory Theorem, $\forall n > N$ there exist $(\lambda_i^n)_{i=1}^{C+1} \in \Delta_{C+1} := \left\{(\alpha_i)_{i=1}^{C+1} \in \mathbb{R}_+^{C+1} : \sum_{i=1}^{C+1} \alpha_i = 1\right\}$ and

$$\left\{u_1^n, ..., u_i^n, ..., u_{C+1}^n\right\} \subseteq \left(N^{>}(x_n) \cap S(0,1)\right) \tag{33}$$

(where the above points are not necessarily distinct) such that

$$v_n = \sum_{i=1}^{C+1} \lambda_i^n u_i^n.$$

Since $S(0,1)$ is compact, up to a subsequence,

$$\text{for any } i \in \{1, ..., C+1\},\ u_i^n \overset{n}{\to} \overline{u}_i \in S(0,1), \tag{34}$$

Then we have that for any $i \in \{1, ..., C+1\}$, $u_i^n \in N^{>}(x_n)$, $u_i^n \overset{n}{\to} \overline{u}_i$ and, by assumption $x_n \overset{n}{\to} \overline{x}$. Since from Proposition 28, which uses Assumption (ii), the set valued function $N^{>}$ is closed we do have

$$\text{for any } i \in \{1, ..., C+1\},\ u_i^n \overset{n}{\to} \overline{u}_i \in N^{>}(\overline{x}),$$

and, then from (34),

$$\text{for any } i \in \{1, ..., C+1\},\ u_i^n \overset{n}{\to} \overline{u}_i \in N^{>}(\overline{x}) \cap S(0,1). \tag{35}$$

Since Δ_{C+1} is compact, up to a subsequence, we have that $(\lambda_i^n)_{i=1}^{C+1} \overset{n}{\to} \left(\overline{\lambda}_i\right)_{i=1}^{C+1} \in \Delta_{C+1}$.
Then

$$v_n = \sum_{i=1}^{C+1} \lambda_i^n u_i^n \overset{n}{\to} \sum_{i=1}^{C+1} \overline{\lambda}_i \overline{u}_i = \overline{v}, \tag{36}$$

where the second equality follows from the assumption that $v_n \overset{n}{\to} \overline{v}$ and uniqueness of limit. From (36) and (35), we have that $\overline{v} \in \mathrm{conv}(N^{>}(\overline{x}) \cap S(0,1)) := G(\overline{x})$, as desired.
Conclusion 2.a follows from Proposition 31.1 and Conclusion 1 above.
Conclusion 2.b follows from Proposition 31.2., Conclusion 2a above. ∎

3 Comparison with the available Variational Inequality results

Several papers analyze the relationship between the solutions to problems $M(K, P)$ and $\mathrm{GVI}(K, P)$ and conditions under which a solution to any of them does exist - see for example, [4], [27] and the papers quoted there. Some of those papers deal with the case in which X is Banach space; they usually also present a specification of those results in the case $X \subseteq \mathbb{R}^C$.
In what follows, we compare our results with the most general available results in the case in which the choice set X is a subset of $\mathbb{R}^C$. The basic conclusions are as follows: our sufficient and necessary conditions are more general than those available in the literature; our existence result in the case of compact K is more general than the same result available in the literature. For the reader's convenience, results used to support our statements are contained in Appendix 6.2
We divide the results available in the literature in two main sets: those dealing with preferences described by utility functions and those by general (not necessarily complete or transitive) preferences.

3.1 Sufficient conditions for maximality

3.1.1 Results in terms of utility functions

Definition 37 *Let A, B be subsets of $\mathbb{R}^C$ and a function $f : A \to \mathbb{R}$ be given. We say that f is radially continuous at $x \in A$ with respect to B if for every $v \in B$, $\lim_{t\to+\infty} f(x + tv) = f(x)$.Results in terms of utility functions*

The main results existing in the literature are presented in Proposition 5.16, page 196, and Proposition 5.14, page193 - which is used in Proposition 5.16 - in [4]. The statement of Proposition 5.16, using our notation and a maximization, not minimization, approach is as follows.

Proposition 38 *Let the function $f : X \subseteq \mathbb{R}^C \to \mathbb{R}$ be given. If*
$\overline{x}$ solves GVI-cone (K, P) (60) and
1 . f is quasiconcave;
2. f is radially continuous on X;
either 3.1. $K \subseteq \mathrm{Int}(X)$;
or 3.2. $\mathrm{aff}(K) = X$ or equivalently (i.2) $K^{\perp} = \{0\}$;[6]*.*
4. $\overline{x}$ is not a global maximum[7]*.*
Then,

$$\overline{x} \quad \textit{solves} \quad \max_{x \in K} f(x).$$

Our Propositions 80 in Appendix 6.2 says what follows. If $\overline{x}$ solves GVI (K, P) and either 1. Assumptions (i.2), (iii),(iv), (v), (vi) or 2. Assumptions (1.2**), (iii), (iv), (v), (vi) are satisfied, then $\overline{x}$ solves $M(K, P)$.
Therefore, from Proposition 79 in Appendix 6.2, our Proposition 80 is strictly more general in terms of assumptions with respect to Proposition 38 (indeed radially continuous implies Assumption (iii) and not vice-versa) and strictly more general in terms of conclusion, since it deals with a maximal and not a maximum result.

3.1.2 Results in terms of general preferences

The main paper on the topic is [27].[8] Conditions presented in Theorem 3b, page 884, are the following ones:K is convex, $K \neq \varnothing$, K is closed and $\succ$ is lower semicontinuous, i.e., for any $x \in X$, $P(x)$ is open.
In their proof, the following statement is not supported by their maintained assumptions:

$$x_n := x' + \frac{1}{n} h \in U(\overline{x}).$$

Indeed, by definition, $U(\overline{x}) \subseteq X$ and $x' + \frac{1}{n}h$ may not belong to X if, for example, x' belong to the boundary of X, but h points outside the set. We do provide a similar argument, but we make an assumption that insures that $x' + \frac{1}{n}h$ does belong to X: indeed that assumption is (i.2**) - see the proof of Proposition 1.19.b.
An example in [4], page 197, shows a case in which
i. all the assumption of Theorem 3b in [27] are satisfied: K is nonempty, convex and closed; $U(x)$ is open (and indeed preferences can be represented by a continuous utility function);
ii. x solves the variational inequality problem, but it is not a maximal element.
Observe that example does not show that our Proposition is false, because that Proposition assumes either (i.2) or (i.2**) and both those assumptions are violated in the example mentioned above. In the Appendix, we present a detailed account of the above statements.

3.2 Necessary conditions for maximality

3.2.1 Results in terms of utility functions

The main results existing in the literature are presented in Proposition 5.18, page 198, in [4]. The statement of that Proposition, using our notation and a maximization, not minimization, approach is as follows.

Proposition 39 *Let the function $f : X \subseteq \mathbb{R}^C \to \mathbb{R}$ be given. If*
1. f is semistrictly quasiconcave;
2. f is continuous;
3. $\mathrm{Int}(\{x \in X : f(x) \geq f(\overline{x})\}) \neq \varnothing$
4. $\overline{x}$ is not a global maximum,
5. K is a convex set (and nonempty)
Then,

$$\overline{x} \textit{ solves } \max_{x \in K} f(x) \Rightarrow \overline{x} \textit{ solves GVI } (K, P).$$

From Corollary 68 in Appendix 6.1, we can substitute Assumption 1. and 4. in the above Proposition with 2'.:f is quasiconcave, and 4'.: i.e., f is locally nonsatiated. Therefore, an equivalent version Proposition 39 is the following one.

Proposition 40 *Let the function $f : X \subseteq \mathbb{R}^C \to \mathbb{R}$ be given. If*
1'. f is quasiconcave;
2. f is continuous;
3. $\mathrm{Int}(\{x \in X : f(x) \geq f(\overline{x})\}) \neq \varnothing$

[6] In Remark 5.2.b, page 197 of [4], the Author says also "An alternative hypothesis was investigated in [5]". That assumption is exactly: $\mathrm{Cl}(\mathrm{aff}(K)) = X$ or equivalently $K^{\perp} = \{0\}$. Recall that in an Euclidean space, $\mathrm{aff}(K)$ is a closed set.
[7] That assumption is not written in the statement of the Proposition, but it is used in the proof, since the proof uses Proposition 5.14
[8] [26] is less general than [27]

4'. f is locally nonsatiated;
5. K is a convex set (and nonempty).
Then,

$$\overline{x} \text{ solves } \max_{x \in K} f(x) \Rightarrow \overline{x} \text{ solves } GVI(K, P).$$

Our Proposition is as follows.

Proposition 41 *If the following assumptions are satisfied*
(iv) P is convex valued;
(viii) (Local NonSatiation)
(i.1) K is convex;
(i.2) $K \neq \varnothing$,*
then

$$\overline{x} \text{ solves } M(K, P) \text{ and } \quad \Rightarrow \quad \overline{x} \text{ solves } GVI\ (K, P).$$

Therefore our results is strictly stronger, simply because we do not need assumptions 2. or 3.

3.2.2 Results in terms of general preferences

The main paper on the topic is [27]. Theorem 3a is the same as ours. Keep in mind that $x \in \mathrm{Cl}_{\mathbb{R}^C}(P(x)) \overset{\text{Rmk } 20}{\Leftrightarrow} P$ is locally non-satiated.

3.3 Existence conditions

3.3.1 Results in terms of utility functions

In the case of K compact, Corollary 4.4, page 10, in [5] says what follows.[9]

Proposition 42 *If $f : X \to \mathbb{R}$ satisfies the following assumptions*
(i.1) K is convex;
(i.2) $K^{\perp} = \{0\} \Rightarrow K \neq \varnothing$;
(i.2) $K \neq \varnothing$;*
(i.3) K is compact;
1. f is quasiconcave;
2. f is radially continuous;
3. f is upper semicontinuous;
5. for every $x \in X$ which is not a global maximum , $\varnothing \neq \mathrm{Int}(\{z \in X : f(z) \geq f(x)\}) \overset{Prop.\ 76}{=} \{z \in X : f(z) > f(x)\}$;
6. $\overline{x}$ is not a global maximum;
then

$$\max_{x \in K} f(x)$$

has a solution.

Our result about existence of a maximal element is as follows.

Proposition 43 *If*
(i.1) K is convex;
(i.2) $K^{\perp} = \{0\}$;
(i.2) $K \neq \varnothing$;*
(i..3) K is compact;
(iv) P is convex valued;
(ii) P is lower semicontinuous;
(iii) (Openness like assumption) For any $x \in X$, $y \in P(x)$ and $z \in X \setminus \{y\}$, we have $[z, y) \cap P(x) \neq \varnothing$;
(vii) For any $x \in X$, there exists $y \in P(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z, y) \cap P(x) \neq \varnothing$;*
(v) (Irreflexivity) For any $x \in X$, $x \notin P(x)$;
(vi) (Global NonSatiation) P is non-empty valued;
then $\max_{x \in X} f(x)$ *has a solution.*

All our assumptions are more general that Assumptions 1-8; observe that Assumption 5. can be restated as $\mathrm{Int}_{\mathbb{R}^C} P(x) \neq \varnothing$, which implies Assumption (vii*).

[9] In the quoted theorem, a coercivity condition replaces compactness of K.

3.3.2 Results in terms of general preferences

The main paper on the topic is [27]. Conditions presented in Theorem 4, page 885, are the following ones: K is convex, $K \neq \varnothing$, K is compact, $\succ$ is upper semicontinuous, P is convex valued, for any $x \in X$, $x \notin P(x)$.[10]
Our result is more general because of Proposition 82 in Appendix 6.2, which says that our assumption on lower semicontinuity of P is strictly more general than assumption of lower semicontinuity of $\succ$ in [27].

4 Existence of equilibria in an exchange economy

4.1 Definition of equilibrium

We consider a pure exchange economy in which $C \in \mathbb{N}$ different commodities or goods, denoted by $c \in \mathcal{C} := \{1, 2, \ldots, C\}$, are traded among $H \in \mathbb{N}$ households or consumers, denoted by $h \in \mathcal{H} := \{1, 2, ..., H\}$. Each consumer $h \in \mathcal{H}$ is described by a consumption space $X_h \subseteq \mathbb{R}^C$, an endowment $e_h \in X_h$ and a set valued preference function $P_h : (\times_{h' \in \mathcal{H}} X_{h'}) \rightrightarrows X_h$, where for any $x \in \times_{h' \in \mathcal{H}} X_{h'}$, $P_h(x)$ is interpreted as the set of consumption vectors in X_h strictly preferred to x_h for given $(x_{h'})_{h' \in \mathcal{H} \setminus \{h\}}$. We define $X = (\times_{h' \in \mathcal{H}} X_{h'})$, $P = \times_{h \in \mathcal{H}} P_h$ and $\mathcal{P}$ as the set of all P.
We denote $x_h^c \in \mathbb{R}$ and $e_h^c \in \mathbb{R}$ as the consumption and the endowment of commodity c of household h, respectively[11]. We define $x_h = (x_h^c)_{c \in \mathcal{C}} \in X_h$, $x = (x_{h'})_{h' \in \mathcal{H}} \in X$ and similarly $e_h = (e_h^c)_{c \in \mathcal{C}} \in X_h$, $e = (e_{h'})_{h' \in \mathcal{H}} \in X$. Moreover, we denote by $p^c \in \mathbb{R}$ the price of commodity c and $p = (p^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$.
We define an economy as an element $\mathcal{E} = (e, P) \in X \times \mathcal{P}$. All the analysis we present applies to any $\mathcal{E}$ satisfying some properties described below.
The budget constraint set valued function is defined as follows. For any $h \in \mathcal{H}$,

$$\beta_h : \mathbb{R}^C \rightrightarrows X_h, \qquad \beta_h(p) = \{x_h \in X_h : \langle p, x_h - e_h \rangle_C \leq 0\}. \tag{37}$$

Definition 44 *A vector $(\widetilde{x}, \widetilde{p}) \in X \times \mathbb{R}^C$ is an equilibrium for the economy $\mathcal{E} \in E$ if*

1. *Given $\mathcal{E} \in E$ and $p \in \mathbb{R}^C$, for any $h \in \mathcal{H}$, $\widetilde{x}_h$ is a a maximal element for P_h on $\beta_h(\widetilde{p})$, i.e.,*

$$\widetilde{x}_h \in \beta_h(\widetilde{p}) \quad \text{and} \quad P_h(\widetilde{x}) \cap \beta_h(\widetilde{p}) = \varnothing; \tag{38}$$

2. *$\widetilde{x}$ satisfies market clearing conditions, i.e.,*

$$\sum_{h \in \mathcal{H}} (\widetilde{x}_h - e_h) = 0. \tag{39}$$

Assumptions. For any $h \in \mathcal{H}$,

(i) X_h is non-empty, closed, convex and bounded below[12];

(ii) P_h is lower semicontinuous;

(iii) (*Openness like assumption*) For any $x \in X$, $y_h \in P_h(x)$ and $z_h \in X_h \setminus \{y_h\}$, we have $[z_h, y_h) \cap P_h(x) \neq \varnothing$.

(iv) P_h is convex valued.

(v) (*Irreflexivity*) For any $x \in X$, $x_h \notin P_h(x)$.

(vi) (*Global Nonsatiation*) For any $x \in X$, there exists $y_h \in X_h$ such that $y_h \in P_h(x)$.

(vii*) For any $x \in X$, there exists $y \in P(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z, y) \cap P(x) \neq \varnothing$;

(vii) $e_h \in \mathrm{Int}_{\mathbb{R}^C}(X_h)$.

Define the so-called augmented preference set-valued function $\widehat{P}_h$ as follows.

$$\widehat{P}_h : X \rightrightarrows X_h,$$

$$x \rightrightarrows \cup_{y_h \in P_h(x)} (x_h, y_h] = \{(1 - \lambda) x_h + \lambda y_h : \ y_h \in P_h(x) \ \text{ and } \ \lambda \in (0, 1]\}.$$

Remark 45 *Augmented preferences were introduced in [16].*

[10] In [27], assumption P nonempty valued is assumed; our proof can be easilty extended to their case.
[11] Given $v, w \in \mathbb{R}^N$, we denote by $v \gg w$, $v \geq w$ and $v > w$ the standard binary relations between vectors. Also the definitions of the sets $\mathbb{R}^N_+$ and $\mathbb{R}^N_{++}$ are the common ones.
[12] $X_h \subseteq \mathbb{R}^C$ is bounded below if there exists $\underline{x}_h \in \mathbb{R}^C$ such that for any $x_h \in X_h$, we have $x_h \geq \underline{x}_h$.

Proposition 46 *For any $h \in \mathcal{H}$, if P_h satisfies Assumptions (ii) - (vii*), then $\widehat{P}_h$ satisfies the following properties.*

(i) For any $x \in X$, $P_h(x) \subseteq \widehat{P}_h(x)$;

(ii) $\widehat{P}_h$ is lower semicontinuous;

(iii) (Openness like assumption) For any $x \in X$, $r_h \in \widehat{P}_h(x)$ and $z_h \in X_h \setminus \{r_h\}$, we have $[z_h, r_h) \cap \widehat{P}_h(x) \neq \varnothing$;

(iv) $\widehat{P}_h$ is convex valued on X;

(v) (Irreflexivity) For any $x \in X$, $x_h \notin P_h(x)$;

(vi) (Global Nonsatiation) For any $x \in X$, $y_h \in X_h$ such that $y_h \in \widehat{P}_h(x)$.

(vii) for any $y_h \in \widehat{P}_h(x)$, $[y_h, x_h) \subseteq \widehat{P}_h(x)$;

(viii) (Local Nonsatiation) For any $x \in X$, $\varepsilon > 0$, there exists $y_h \in \mathcal{B}(x_h, \varepsilon)$ such that $y_h \in \widehat{P}_h(x)$.

(vii) For any $x \in X$, there exists $y \in \widehat{P}(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z, y) \cap \widehat{P}(x) \neq \varnothing$;*

Proof. For a detailed proof of (i) - (vii) see Proposition 4.1, page 27 in [1]. The proof of (viii) is obvious. The proof of (vii*) is obvious as well, and we present it below just for completeness. We assume that For any $x \in X$, there exists $y \in P(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z, y) \cap P(x) \neq \varnothing$; we want to show that For any $x \in X$, there exists $y' \in \widehat{P}(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y'\}$, $[z, y') \cap \widehat{P}(x) \neq \varnothing$. Take $y' = y \in P(x) \overset{(i)}{\subseteq} \widehat{P}(x)$. Then, for any $z \in \mathbb{R}^C \setminus \{y\}$, $\varnothing \neq [z, y') \cap P(x) \overset{(i)}{\subseteq} [z, y') \cap \widehat{P}(x)$. ■

It is easy to see that "equilibria in terms of $\widehat{P}$" are "equilibria for the true economy with preferences described by P", as formalized below.

Proposition 47 *If $(\widetilde{x}, \widetilde{p})$ is an equilibrium for the economy $\left(e, \widehat{P}\right)$, then $(\widetilde{x}, \widetilde{p})$ is an equilibrium for the economy (e, P).*

Proof. We have to prove that for any $h \in \mathcal{H}$,

$$P_h(\widetilde{x}) \cap \beta_h(\widetilde{p}) = \varnothing,$$

which is obvious because, by assumption,

$$\widehat{P}_h(\widetilde{x}) \cap \beta_h(\widetilde{p}) = \varnothing$$

and from Proposition 46.i, we have $P_h(\widetilde{x}) \subseteq \widehat{P}_h(\widetilde{x})$. ■

As a consequence of Proposition 47, to show existence of an equilibrium for an economy (e, P), it suffices to show existence of an equilibrium for an economy $\left(e, \widehat{P}\right)$ which is what is done in the remainder of the Section.

4.2 Definition of the budget set valued function and its properties

Using a standard trick, in this subsection we define a fictitious budget set-valued function in which, for any $h \in \mathcal{H}$, we add the constraint $x_h \leq r' := r + M \cdot \mathbf{1}_G$ where $r := \sum_{h \in \mathcal{H}} \left(|e_h^c| + |\underline{x}_h^c|\right)_{c \in \mathcal{C}}$ and $M > 0$; we first show existence with the "artificial bound r'" and then without it.

We restrict our search for equilibrium prices to the closed unit ball $B := \left\{p \in \mathbb{R}^C : \|p\| \leq 1\right\}$. To avoid problems in showing needed properties of the budget set-valued function, we further modify it adding the extra term $1 - \|p\|$ to the right hand side of the inequality that defines it. Bergstrom was the first author to use this trick - see [9].

Proposition 48 *For any $h \in \mathcal{H}$, the set-valued function*

$$\beta_h' : B \to X_h, \quad \beta_h'(p) = \{x_h \in X_h : \langle p, x_h - e_h \rangle_C \leq 1 - \|p\| \ \text{ and } x_h \leq r'\}$$

is

1. *nonempty valued and convex valued;*
2. *closed;*
3. *compact valued;*
4. *lower semicontinuous;*
5. *upper semicontinuous.*

Proof. 1.
For any $p \in B$, take $x_h = e_h$. Convexity follows from the fact that the constraint functions are affine and X_h is convex.
2.
Let $(p_n)_{n\in\mathbb{N}} \in B^\infty$ be a sequence such that $\lim_{n\to+\infty} p_n = p$ and let $(x_{h,n})_{n\in\mathbb{N}} \in (X_h)^\infty$ be such that $x_{h,n} \in \beta'_h(p_n)$ and $\lim_{n\to+\infty} x_{h,n} = x_h$. We want to show that $x_h \in \beta'_h(p)$. First of all, observe that since X_h is closed from Assumption (i), then $x_h \in X_h$. Moreover, we have that for any $n \in \mathbb{N}$,

$$x_{h,n} \leq r'$$

$$\langle p_n, x_{h,n} - e_h\rangle_C \leq 1 - \|p_n\|$$

Then, all the functions involved in the left hand sides of the above inequalities are continuous and taking limits we get that $x_h \in \beta'_h(p)$, as desired.
3.
For any $p \in B$, $\beta'_h(p)$ is closed because defined in terms of continuous functions and weak inequalities. $\beta'_h(p)$ is bounded below by Assumption (i) and bounded above by r'.
4.
To prove the lower semicontinuity of β'_h, we define the set-valued function

$$\beta''_h : B \rightrightarrows X_h,$$

$$\beta''_h(p) = \{x_h \in X_h : x_h \in \operatorname{Int}(X_h), \quad x_h << r', \text{ and } \langle p, x_h - e_h\rangle_C < 1 - \|p\| .\}$$

First of all, let's show that

$$\beta''_h \text{ is non-empty valued.} \tag{40}$$

If $p = 0$, then $e_h \in \operatorname{Int}(X_h)$, from Assumption (vii), $e_h \leq \sum_{h'\in\mathcal{H}} \left((|e^c_{h'}|)_{c\in C}\right) << r'$and $0 = \langle p, e_h - e_h\rangle_C < 1-\|p\| = 1$. Now suppose that $p \neq 0$; observe that there exists $\varepsilon > 0$ such that $\mathcal{B}(e_h, \varepsilon) \subseteq X_h$, because $e_h \in \operatorname{Int}(X_h)$, from Assumption (vii). Define $x^*_h = (x^{*c}_h)_{c\in\mathcal{C}}$ such that

$$x^{*c}_h = \begin{cases} e^c_h - \frac{1}{n} & \text{if} \quad p^c > 0, \\ e^c_h & \text{if} \quad p^c = 0, \\ e^c_h + \frac{1}{n} & \text{if} \quad p^c < 0, \end{cases}$$

with $n \in \mathbb{N}$ large enough so that $x^*_h \in \mathcal{B}(e_h, \varepsilon) \subseteq X_h$ and $x^*_h << r'$. Then, $x^*_h \in \operatorname{Int}(X_h)$, $x^*_h << r'$, and

$$\langle p, x_h - e_h\rangle_C = -\frac{1}{n}\sum_{c\in\mathcal{C}} |p^c| < 0 \leq 1 - \|p\| .$$

Now to show that β'_h is lower semicontinuous, we go through 3 steps. Step 1. β'_h is the closure of β''_h. Step 2. β''_h is lower semicontinuous. Step 3. Desired result.
Step 1. $\beta'_h = \operatorname{Cl}(\beta''_h)$.
We go through three substeps: i. for any $p \in B$

$$\operatorname{Int}(\beta'_h(p)) = \beta''_h(p) \overset{(40)}{\neq} \varnothing; \tag{41}$$

ii. $\operatorname{Cl}(\operatorname{Int}(\beta'_h(p))) = \operatorname{Cl}(\beta'_h(p))$;
iii. desired result.
i.
$\cdot$ $\beta''_h(p) \subseteq \operatorname{Int}(\beta'^{(p)}_h)$
By definition, $\beta''_h(p) \subseteq \beta'_h(p)$ and then $\operatorname{Int}(\beta''_h(p)) \subseteq \operatorname{Int}(\beta'_h(p))$. Since $\beta''_h(p)$ is an open set, then $\beta''_h(p) = \operatorname{Int}(\beta''_h(p))$. Hence $\beta''_h(p) \subseteq \operatorname{Int}(\beta'_h(p))$.
$\cdot\cdot$ $\operatorname{Int}(\beta'_h) \subseteq \beta''_h$.
Take $\widehat{x}_h \in \operatorname{Int}(\beta'_h(p))$. Then, there exists $\delta > 0$ such that $\mathcal{B}(\widehat{x}_h, \delta) \subseteq \beta'_h(p) \subseteq X_h$ and then $\widehat{x}_h \in \operatorname{Int}(X_h)$. Suppose now our desired result does not hold true, i.e., $\widehat{x}_h \notin \beta''_h(p)$, i.e., it is not the case that

$$\widehat{x}_h << r' \qquad \text{or} \qquad \langle p, \widehat{x}_h - e_h\rangle_C < 1 - \|p\| .$$

Since $\widehat{x}_h \in \operatorname{Int}(\beta'_h(p)) \subseteq \beta'_h(p)$, we have that either $\exists\ c \in \mathcal{C}$ such that $\widehat{x}^c_h = (r')^c$ or $\langle p, x_h - e_h\rangle_C = 1 - \|p\|$.
If $\exists\ c \in \mathcal{C}$ such that $\widehat{x}^c_h = (r')^c$, then take $\widehat{\widehat{x}}_h$ such that

$$\widehat{\widehat{x}}^{c'}_h = \begin{cases} (r')^{c'} & \text{if} \quad c' \neq c, \\ \\ (r')^c + \frac{1}{n} & \text{if} \quad c' = c. \end{cases}$$

Since $\widehat{x}_h \in \mathrm{Int}(\beta'_h(p))$, for n large enough, $\widehat{\widehat{x}}_h \in \beta'_h(p)$; but $\widehat{\widehat{x}}_h^{c'} > (r')^c$, contradicting the definition of β'_h.
If $\langle p, \widehat{x}_h - e_h\rangle_C = 1 - \|p\|$, we get a contradiction as well. Indeed, if $p = 0$, then we get $0 = 1$. If $p \neq 0$, then take $\widehat{x}_h + \frac{1}{n}(\text{sign } p^c)_{c\in\mathcal{C}}$, which for n large enough belongs to $\mathcal{B}(\widehat{x}_h, \delta) \subseteq \beta'_h(p)$. Then,

$$1 - \|p\| \geq \langle p, \widehat{x}_h + \frac{1}{n}\left((\text{sign } p^c)_{c\in\mathcal{C}}\right) - e_h\rangle_C = 1 - \|p\| + \sum_{c\in\mathcal{C}} \frac{1}{n}|p^c| > 1 - \|p\|,$$

a contradiction.
ii.
It follows from Proposition 61 in the Appendix.
iii.
Observe that $\beta'_h(p)$ is closed (recall that X_h is closed) and that $\beta''_h(p)$ is open valued because defined in $\mathrm{Int}(X_h)$ and in terms of strict inequalities. Then

$$\varnothing \overset{(41)}{\neq} \mathrm{Int}\beta'_h(p) = \beta''_h(p) = \mathrm{Int}\left(\beta''_h(p)\right). \tag{42}$$

Then, we can apply again apply the result mentioned in ii. above to get

$$\mathrm{Cl}\left(\beta''_h(p)\right) \overset{(42)}{=} \mathrm{Cl}\left(\mathrm{Int}\beta'_h(p)\right) \overset{(1)}{=} \beta'_h(p),$$

where (1) follows from the fact that $\beta'_h(p)$ is closed.
Step 2. β''_h is lower semicontinuous
$\beta''_h(p)$ is not empty from (41). For every sequence $(p_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} p_n = p$ let $x_h \in \beta''_h(p)$. Then,

$$\lim_{n\to+\infty}\Big(\langle p_n, x_h - e_h\rangle_C - (1 - \|p_n\|) = \Big(\langle p, x_h - e_h\rangle_C - (1 - \|p\|) < 0,$$

and

$$\lim_{n\to+\infty} x_h - r' = x - r' << 0,$$

Therefore, there exists $\nu \in \mathbb{N}$ such that for all $n > \nu$, we have $x_h \in \beta''_h(p_n)$. Then, we can choose the sequence $(x_{h,n})_{n\in\mathbb{N}}$ as $x_{h,n} = x_h$and we can conclude that β''_h is lower semicontinuous.
Step 3. β'_h is lower semicontinuous.
Since β'_h is the closure of a lower semicontinuous set valued function, then the desired result follows from Proposition 69 in the Appendix.
5.
It follows from the four results above and Proposition 71 in the Appendix. ■

4.3 The Variational Inequality problem

In this section we introduce the variational inequality problem which allows us to show existence of equilibria.
First of all, consider the set-valued function

$$G_h : X_h \rightrightarrows X_h, \qquad G_h(x_h) = conv\Big(N_h^{>}(x_h) \cap S(0,1)\Big)$$

where $S(0,1) = \{x \in \mathbb{R}^C : \|x\|_C = 1\}$ is the unit sphere of $\mathbb{R}^C$ and $N_h^{>}(x_h)$ is the normal cone to $P_h(x)$. Define also the set valued map $\beta' = \times_{h\in\mathcal{H}}\beta'_h$.
We introduce the following variational problem

$$\begin{aligned} &\text{Find } (\widetilde{x}, \widetilde{p}) \in \beta'(\widetilde{p}) \times B \text{ and } g = (g_h)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} G_h(\widetilde{x}_h) \text{ such that} \\ &\text{for any } (x,p) \in \beta'(\widetilde{p}) \times B, \qquad \textstyle\sum_{h\in\mathcal{H}} \langle -g_h, x_h - \widetilde{x}_h\rangle_C + \langle \sum_{h\in\mathcal{H}}(\widetilde{x}_h - e_h)), (p - \widetilde{p}))\rangle_C \leq 0. \end{aligned} \tag{43}$$

Remark 49 *$(\widetilde{x}, \widetilde{p}) \in \beta'(\widetilde{p}) \times B$ is a solution to (43) if and only if, each of the following statements holds true,.*
1. for any $h \in \mathcal{H}$, there exists $\widetilde{x}_h \in \beta'_h(\widetilde{p})$ and $g_h \in G_h(\widetilde{x}_h)$ such that

$$\langle -g_{h,}, x_h - \widetilde{x}_h\rangle_G \leq 0 \quad \forall x_h \in \beta'_h(\widetilde{p}); \tag{44}$$

2.

$$\langle \sum_{h\in\mathcal{H}}(\widetilde{x}_h - e_h)), (p - \widetilde{p}))\rangle_C \leq 0,\ \forall p \in B. \tag{45}$$

In our analysis, we are going to us the following Variational Inequality problem and the related existence result presented in Theorem 78 in the Appendix.

A *Generalized Quasi-Variational Inequality* associated with K, F, denoted by GQVI, is the following problem:

$$\text{“Find } \overline{x} \in K(\overline{x}) \text{ and } \overline{u} \in F(\overline{x}) \text{ such that for any } x \in K(\overline{x}),\ \ \langle \overline{u}, x - \overline{x} \rangle \geq 0.\text{”} \tag{46}$$

Theorem 50 *The variational problem (43) admits at least one solution.*

Proof. To get the desired result, we apply Theorem 78 . We define C, K and F and we check the desired assumptions are satisfied.

$\diamond$ $C := conv(\beta'(B)) \times B \subseteq \mathbb{R}^{CH} \times \mathbb{R}^{C} := \mathbb{R}^{l}$ *is nonempty, convex and compact.*

B is nonempty and compact; from Proposition 48, we have that β' is nonempty, convex and compact valued, closed and lower semicontinuous and upper semicontinuous on B and $conv(\beta'(B)$ is (nonempty) convex and compact as well. In the above argument, we use Propositions 70, 71 and 72 in the Appendix.

$\diamond$ *The set-valued function*

$$K : C \rightrightarrows \mathbb{R}^{CH} \times \mathbb{R}^{C}, \qquad K(x,p) = \beta'(p) \times B.$$

is nonempty, convex, compact valued, closed, lower semicontinuous and upper semicontinuous.

It follows from what said above.

$\diamond$ T*he set-valued function*

$$F : C \rightrightarrows \mathbb{R}^{CH} \times \mathbb{R}^{C}, \qquad F(x,p) = \times_{h\in\mathcal{H}}(G_h(x_h) \times \left\{ -\sum_{h\in\mathcal{H}} (x_h - e_h)) \right\}.$$

is nonempty, convex, compact valued, closed and upper semicontinuous.

The set-valued function G_h is basically equal to the set-valued function G, presented in Definition 29: the fact that the domain of $\widehat{P}_h$ is X and not X_h does not change of the results obtained there. Observe that $\widehat{P}_h$ satisfies all the properties listed in Proposition 46. Moreover, $e_h \in \beta'_h(p) \cap (\mathrm{Int}_{\mathbb{R}^C}(X_h))$. Then, from the proof of Proposition 36, we have that for any $h \in H$, G_h is nonempty, convex, compact valued, closed and upper semicontinuous. Furthermore the other components of F are continuous functions. Then all assumptions of Theorem 78 are satisfied and we can conclude that

$$\begin{array}{l} \exists \quad (\widetilde{x}, \widetilde{p}) \in \beta'(\widetilde{p}) \times B \quad \text{and} \quad (\widetilde{g}, \widetilde{r}) \in \times_{h\in\mathcal{H}}(G_h(\widetilde{x}_h) \times \left\{ -\sum_{h\in\mathcal{H}}(\widetilde{x}_h - e_h)) \right\} \\ \text{such that} \\ \forall (x,p) \in \beta'(\widetilde{p}) \times B, \qquad \langle (\widetilde{g}, \widetilde{r}), (x,p) - (\widetilde{x}, \widetilde{p}) \rangle \geq 0, \end{array}$$

or

$$\begin{array}{l} \exists\, \widetilde{p} \in B \quad \text{and} \quad \forall h \in \mathcal{H} \ \ \exists\, \widetilde{x} \in \beta'(\widetilde{p}) \quad \text{and} \quad \widetilde{g} \in \times_{h\in\mathcal{H}}(G_h(\widetilde{x}_h) \\ \text{such that} \\ \forall p \in B, \quad \forall h \in \mathcal{H}\ \forall x_h \in \beta'_h(\widetilde{p}) \ \ \sum_{h\in\mathcal{H}} \langle g_h, x_h - \widetilde{x}_h \rangle_C + \langle -\sum_{h\in\mathcal{H}}(\widetilde{x}_h - e_h)), (p - \widetilde{p})) \rangle_C \geq 0, \end{array}$$

i.e., the Variational Inequality problem (43) admits at least a solution. ■

4.4 Existence of equilibrium

Theorem 51 *Let*

$$(\widetilde{x}, \widetilde{p}, g) \in \beta'(\widetilde{p}) \times B \times (\times_{h\in\mathcal{H}}(G_h(\widetilde{x}_h))$$

be a solution to the Variational Inequality Problem (43). Then, the following results hold true.

1. for any $h \in \mathcal{H}$, $\widetilde{x}_h$ is optimal in $\beta'_h(\widetilde{p})$, i.e.,

$$\begin{array}{l} \widetilde{x}_h \in \beta'_h((\widetilde{p}) \text{ and} \\ \widehat{P}_h(\widetilde{x}) \cap \beta'_h((\widetilde{p}) = \varnothing; \end{array} \tag{47}$$

2. $\sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h) = 0$;

3. Budget constraints hold true as equalities, i.e., for any $h \in \mathcal{H}$

$$\langle \widetilde{p}, \widetilde{x}_h - e_h \rangle_C = 1 - \|p\|,$$

4. $\|\widetilde{p}\| = 1$.

Proof. 1.

By assumption and from Remark 49 for any $h \in \mathcal{H}$, there exists $\widetilde{x}_h \in \beta'_h(\widetilde{p})$ and $g_h \in G_h(\widetilde{x}_h)$ such that for any $x_h \in \beta'_h(\widetilde{p})$, we have $\langle -g_h, x_h - \widetilde{x}_h \rangle_G \leq 0$. Moreover, from (41), we have that for any $p \in B$, $\mathrm{Int}\beta'_h(p) \neq \varnothing$ and therefore from Proposition 60 in the Appendix, we have $(\beta'_h(\widetilde{p}))^{\perp} = \{0\}$. From Proposition 48, $\beta'_h(\widetilde{p})$ is nonempty, convex and compact. Summarizing $\beta'_h(\widetilde{p})$ satisfies assumptions (i.1), (i.2), (i.2*), (i.3), and from Proposition 46, $\widehat{P}_h$

satisfies assumptions (ii), (iii), (iv), (v), (vii*): therefore, we can apply Proposition 36.2.a, identifying K, P there with $\beta_h'(\widetilde{p}), \widehat{P}_h$ here to conclude that $\widetilde{x}_h$ is a solution to $M\left(\beta_h'(\widetilde{p}), \widehat{P}_h\right)$, as desired.

2. Define

$$\widetilde{z} = \sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h) .$$

Then, from (45),

$$\text{for any } p \in B, \quad \langle p, \widetilde{z}\rangle_C \leq \langle \widetilde{p}, \widetilde{z}\rangle \leq H \cdot (1 - \|\widetilde{p}\|) , \tag{48}$$

where last inequality follows from the fact that for any $h \in \mathcal{H}$, $\widetilde{x}_h \in \beta'(\widetilde{p})$.
Now suppose our claim is false, i.e., $\widetilde{z} \neq 0$. Then,

$$\text{for any } p \in B, \quad \langle p, \frac{\widetilde{z}}{\|\widetilde{z}\|}\rangle_C \leq \langle \widetilde{p}, \frac{\widetilde{z}}{\|\widetilde{z}\|}\rangle,$$

i.e., $\widetilde{p} \in B$ solves the problem

$$\text{for given } \frac{\widetilde{z}}{\|\widetilde{z}\|} \in B, \;\; \max_{p\in B} \;\; p\frac{\widetilde{z}}{\|\widetilde{z}\|}.$$

From Cauchy-Schwarz inequality,$p\frac{\widetilde{z}}{\|\widetilde{z}\|} \leq \|p\| \cdot \left\|\frac{\widetilde{z}}{\|\widetilde{z}\|}\right\| \leq 1$ and $\frac{\widetilde{z}}{\|\widetilde{z}\|} \cdot \frac{\widetilde{z}}{\|\widetilde{z}\|} = 1$; then, we have $\widetilde{p} = \frac{\widetilde{z}}{\|\widetilde{z}\|}$ and $\|\widetilde{p}\| = 1$. Then, from (48),

$$0 = H \cdot (1 - \|\widetilde{p}\|) \geq \langle \widetilde{p}, \widetilde{z}\rangle = \left\langle \frac{\widetilde{z}}{\|\widetilde{z}\|}, \widetilde{z}\right\rangle = \|\widetilde{z}\| > 0,$$

which is the desired contradiction.

3.

Suppose our claim is false and there exists $h \in \mathcal{H}$ such that

$$\langle \widetilde{p}, \widetilde{x}_h - e_h\rangle_C < 1 - \|p\| . \tag{49}$$

By Assumption (viii), i.e., Local NonSatiation, there exists $x_h \in X_h$ such that $[x_h, \widetilde{x}_h) \subseteq \widehat{P}_h(\widetilde{x})$. From (49), we can choose $x_h' \in [x_h, \widetilde{x}_h)$ such that $x_h' \in \beta_h'(\widetilde{p})$. Then $x_h' \in \beta_h'(\widetilde{p}) \cap \widehat{P}_h(\widetilde{x})$, contradicting part 1. above.

4.

$$0 \overset{2.\text{ above}}{=} \widetilde{p}\widetilde{z} \overset{3.\text{ above}}{=} H \cdot (1 - \|\widetilde{p}\|) ,$$

as desired. ■

Theorem 52 *For any economy* $(e, \widehat{P})$ *with the upper bound* r'*, there exists an equilibrium vector*

$$(\widetilde{x}, \widetilde{p}) \in \beta'(\widetilde{p}) \times B.$$

Proof. From Theorem 51, $\widetilde{x}$ satisfies market clearing and since $\|\widetilde{p}\| = 1$ also the the budget constraints of each households. ■

Proposition 53 *Under Assumption from (i) to (vii), and (vii**), for any economy* $(e, P) \in X \times \mathcal{P}$ *an equilibrium exists.*

Proof. We are going to show that $(\widetilde{x}, \widetilde{p})$ is an equilibrium (without bound). Indeed, we are left with showing that for any $h \in \mathcal{H}$,

$$\widetilde{x}_h \in \beta_h(\widetilde{p}) = \{x_h \in X_h : \langle p, x_h - e_h\rangle_C \leq 0\} \quad \text{and} \quad P_h(\widetilde{x}) \cap \beta_h(\widetilde{p}) = \varnothing; \tag{50}$$

Below we present a proof using the variational inequality problem; a proof can be also provided directly in terms of the maximal problem $M(K, P)$.
First of all, observe that $\widetilde{x}_h \in \beta_h'(\widetilde{p}) \subseteq \beta_h(\widetilde{p})$. For any $h \in \mathcal{H}$, by assumptions, we have that $\exists\, g_h \in G_h(\overline{x}_h)$ such that

$$\forall x_h \in \beta_h'(\widetilde{p}), \quad \langle -g_h, x_h - \widetilde{x}_h\rangle_C \leq 0. \tag{51}$$

We want to prove that

$$\forall x_h \in \beta_h(\widetilde{p}), \quad \langle -g_h, x_h - \widetilde{x}_h\rangle_C \leq 0.$$

Suppose otherwise, i.e., $\exists x_h \in \beta_h(\widetilde{p})$ such that

$$\langle -g_h, x_h - \widetilde{x}_h\rangle_C > 0, \tag{52}$$

and therefore

$$x_h \neq \widetilde{x}_h \tag{53}$$

Since $x_h \in \beta_h(\widetilde{p})$, and β_h is convex, then $\forall \lambda \in (0,1)$, we have $\widehat{x}_h = (1-\lambda)\widetilde{x}_h + \lambda x_h \in \beta_h(\widetilde{p})$. Moreover,

$$\langle -g_h, \widehat{x}_h - \widetilde{x}_h \rangle_C = \lambda \langle -g_h, x_h - \widetilde{x}_h \rangle_C > 0. \tag{54}$$

Now, for any $\delta > 0$, $\|\widehat{x}_h - \widetilde{x}_h\| = \lambda \|x_h - \widetilde{x}_h\| < \delta \Leftrightarrow \lambda < \frac{\delta}{\|x_h - \widetilde{x}_h\|}$. Then, choosing $\lambda < \min\left\{1, \frac{\delta}{\|x_h - \widetilde{x}_h\|}\right\}$, we have that

$$\widehat{x}_h \in \beta(\widetilde{p}) \cap \mathcal{B}(\widetilde{x}_h, \delta). \tag{55}$$

Recall that

$$\beta_h'(\widetilde{p}) := \beta_h((\widetilde{p}) \cap [\underline{x}_h, r'], \tag{56}$$

where $\underline{x}_h$ is the lower bound for X_h. Moreover, from market clearing, $\widetilde{x}_h = \sum_{h' \in \mathcal{H}} \widetilde{e}_{h'} - \sum_{h' \in \mathcal{H}\setminus\{h\}} \widetilde{x}_{h'} \leq \sum_{h' \in \mathcal{H}} \widetilde{e}_{h'} - \sum_{h' \in \mathcal{H}\setminus\{h\}} \underline{x}_{h'} << r'$, where last inequality follows from the definition of r'. Then,

$$\text{there exists } \delta > 0 \text{ such that } \mathcal{B}(\widetilde{x}_h, \delta) \cap X_h \subseteq [\underline{x}_h, r']. \tag{57}$$

Then

$$\beta_h(\widetilde{p}) \cap \mathcal{B}(\widetilde{x}_h, \delta) \overset{\beta_h(\widetilde{p}) \subseteq X_h}{\subseteq} \beta_h(\widetilde{p}) \cap \mathcal{B}(\widetilde{x}_h, \delta) \cap X_h \overset{(57)}{\subseteq} \beta_h(\widetilde{p}) \cap [\underline{x}_h, r'] \overset{(56)}{=} \beta_h'(\widetilde{p}). \tag{58}$$

From (55) and (58),

$$\widehat{x}_h \in \beta_h'(\widetilde{p}). \tag{59}$$

Then (54) and (59) contradict (51). ■

5 Comparison with the available exchange economy results

5.1 Variational Inequality literature

5.1.1 Results in terms of utility functions

To the best of our knowledge, the most general result on existence of equilibria for an exchange economy using a Variational Inequality approach is the one contained in [14][13]. Using our notation, the existence result presented there is as follows.

Theorem 54 *Assume that preferences of consumer $h \in \mathcal{H}$ are represented by an utility function $u_h : X_h \to \mathbb{R}$ and that for any $h \in \mathcal{H}$,*
0. X_h is a closed, convex, bounded below subset of $\mathbb{R}^C$,
1. u_h is continuous,
2. u_h has no global maximum,
3. u_h is semistrictly quasi concave,
Assumption (i) and (vii) hold true.
Then an equilibrium exists.

Indeed, their proof quotes a result in [6] in which $X_h = \mathbb{R}^C$. In any case, from Proposition 83, our result is more general than the above one.

5.1.2 Results in terms of general preferences

The main paper on the topic is [27].[14]
The assumptions of Theorem 8 in [27] are what follows:
for any $h \in \mathcal{H}$

(i) $X_h = \mathbb{R}^C_+$;

(ii) $\succ_h$ is lower and upper semicontinuous;

(iv) $\succ_h$ is convex valued.

(v) (*Irreflexivity*) For any $x \in X$, $x_h \notin P_h(x)$.

[13] Indeed, they analyze a model with production, an extension which can be analyzed using the approach we follow.
[14] [25] is less general than [27].

(vi) (*Global Nonsatiation*) For any $x \in X$, there exists $y_h \in X_h$ such that $y_h \in P_h(x)$.

(vii) $x_h \in \mathrm{Cl}_{\mathbb{R}^C}(P(x_h))$, i.e., local nonsatiation.

The Assumptions in Proposition 36 are what follows:

For any $h \in \mathcal{H}$,

(i) X_h is non-empty, closed, convex and bounded below;

(ii) P_h is lower semicontinuous;

(iii) (*Openness like assumption*) For any $x \in X$, $y_h \in P_h(x)$ and $z_h \in X_h \setminus \{y_h\}$, we have $[z_h, y_h) \cap P_h(x) \neq \varnothing$.

(iv) P_h is convex valued.

(v) (*Irreflexivity*) For any $x \in X$, $x_h \notin P_h(x)$.

(vi) (*Global Nonsatiation*) For any $x \in X$, there exists $y_h \in X_h$ such that $y_h \in P_h(x)$.

(vii*) For any $x \in X$, there exists $y \in P(x)$ such that for any $z \in \mathbb{R}^C \setminus \{y\}$, $[z, y) \cap P(x) \neq \varnothing$;

(vii) $e_h \in \mathrm{Int}_{\mathbb{R}^C}(X_h)$.

In [27] assumption (vii) is not assumed, but it refer to [25] and [26], in which assumption (vii) is made. In any case, our proof is strictly more general than their proof because of the following facts.
1. Lower semicontinuity of $\succ_h$, i.e., P is open valued, implies Assumption (iii), but not viceversa - see Proposition 84 in Appendix 6.2.
2. Upper semicontinuity of $\succ_h$ implies Assumption (ii), but not viceversa - see Proposition 82 in Appendix 6.2
3. Our assumption (i) implies their assumption (i) and not viceversa.
4. We do not assume local nonsatiation.
5. Theorem 8 of [27] does not contain Assumption (i.2**) which should be used to get statement $x_n = x' + \frac{1}{n}h \in U(\overline{x})$, as we observed in Section 3.2.

5.2 Economic literature

Below we present the statements of some general existence results which are available in the economic literature and we compare those results with ours.

Proposition 55 *([8]) An equilibrium exists if the following Assumptions are satisfied.*

For any $h \in \mathcal{H}$,

(i) X_h is non-empty, closed, convex and bounded below;

(iii B) For any $x \in X$, $P^{-1}(x) := \{x' \in X : x \in P(x')\}$ is open.

(iv new) For any $x \in X$, $x_h \notin \mathrm{conv}(P_h(x))$.

(vi) (*Global Nonsatiation*) For any $x \in X$, there exists $y_h \in X_h$ such that $y_h \in P_h(x)$.

(vii) $e_h \in \mathrm{Int}_{\mathbb{R}^C}(X_h)$.

From Proposition 84 in Appendix 6.2 our result and Bergstrom's are independent.

Proposition 56 *([16] and [17]) An equilibrium exists if the following Assumptions are satisfied.*
For any $h \in \mathcal{H}$,

(i) X_h is non-empty, closed, convex and bounded below[15];

(ii) P_h is lower semicontinuous;

(iii minus) (*Openness assumption*) P_h is open valued;

(iv) P_h is convex valued.

(v) (*Irreflexivity*) For any $x \in X$, $x_h \notin P_h(x)$.

[15] $X_h \subseteq \mathbb{R}^C$ is bounded below if there exists $\underline{x}_h \in \mathbb{R}^C$ such that for any $x_h \in X_h$, we have $x_h \geq \underline{x}_h$.

(vi) (*Global Nonsatiation*) For any $x \in X$, there exists $y_h \in X_h$ such that $y_h \in P_h(x)$.

(vii) $e_h \in \mathrm{Int}_{\mathbb{R}^C}(X_h)$.

Remark 57 *The above result is strictly less general than ours because (iii minus) implies (iii) and (vii*) - and not vice versa.*

Proposition 58 *([18]) An equilibrium exists if the following Assumptions are satisfied.*

For any $h \in \mathcal{H}$,

(i) X_h is non-empty, closed, convex and bounded below;

(ii) P_h is lower semicontinuous;

(iii) (*Openness like assumption*) For any $x \in X$, $y_h \in P_h(x)$ and $z_h \in X_h \setminus \{y_h\}$, we have $[z_h, y_h) \cap P_h(x) \neq \varnothing$.

(iv new) For any $x \in X$, $x_h \notin \mathrm{conv}(P_h(x))$.

(vi) (*Global Nonsatiation*) For any $x \in X$, there exists $y_h \in X_h$ such that $y_h \in P_h(x)$.

(vii) $e_h \in \mathrm{Int}_{\mathbb{R}^C}(X_h)$.

Remark 59 *The above result is strictly more general than ours because (iv) and (v) imply (iv new) - and not vice versa.*

6 Appendix.

6.1 Some needed results on convexity and set valued functions

Proposition 60 $\mathrm{Int}_{\mathbb{R}^C}(K) \neq \varnothing \Rightarrow K^{\perp} = \{0\}$.

Proof. We want to show that if $z \in \mathbb{R}^C \setminus \{0\}$, then $z \notin K^{\perp}$, i.e., there exist $x, y \in K$ such that $z(x-y) \neq 0$. By assumption, there exists $x^* \in K$ and $r > 0$ such that $\mathcal{B}(x^*, r) \subseteq K$. Then there exists $\lambda > 0$ and sufficiently small such that $x := x^* + \lambda z$ and $y := x^* - \lambda z$ belong to $\mathcal{B}(x^*, r) \subseteq K$. Then,

$$\langle z, x-y \rangle = \langle z, x^* + \lambda z - (x^* - \lambda z) \rangle = 2\lambda \langle z, z \rangle > 0,$$

as desired. ■

Proposition 61 *(Corollary 2.3.9, page 64, in [32])If K is a convex subset of $\mathbb{R}^n$ such that* $\mathrm{Int}(K) \neq \varnothing$, *then* $\mathrm{Cl}(\mathrm{Int}\ K) = \mathrm{Cl}\ K$.

Proposition 62 *(Theorem 2.4.10, page 70 in [32]) Let A and B be nonempty convex sets in $\mathbb{R}^C$. If $A \cap B = \varnothing$, then there exists $h \in \mathbb{R}^C \setminus \{0\}$ such that*

$$\forall a \in A, \quad \forall b \in B, \quad \langle h, a \rangle \leq \langle h, b \rangle .$$

Proposition 63 *(Corollary 2.5.5, page 76 in [32]) Let A be a nonempty set of $\mathbb{R}^C$. A is a convex cone if and only if for any $a, b \in A$ and any scalar $\lambda, \mu \geq 0, \lambda a + \mu b \in A$.*

Proposition 64 *If $\varphi : X \rightrightarrows Y$ is non-empty valued and is open graph, i.e.,*

$$\textit{graph } \varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\} \textit{ is } ((X, d) \times (Y, d'))\textit{-open},$$

then φ is lower semicontinuous.

Proof. We are going to show that for any $x \in X$ and for every sequence $(x_n)_{n \in \mathbb{N}} \in X^{\infty}$ such that $x_n \to x$, and for every $y \in \varphi(x)$, there exists a sequence $(y_n)_{n \in \mathbb{N}} \in Y^{\infty}$ such that $\forall\ n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ and $y_n \to y$. Since by assumption $y \in \varphi(x)$, we have that $(x, y) \in$ *graph* φ. Therefore, still by assumption there exist $r_x, r_y \in \mathbb{R}_{++}$ such that $\mathcal{B}_{(X,d)}(x, r_x) \times \mathcal{B}_{(Y,d')}(y, r_y) \subseteq$ *graph* φ. Then, since $x_n \to x$, there exists $N_x \in \mathbb{N}$ such that for any $n > N_x$, $x_n \in \mathcal{B}_{(X,d)}(x, r_x)$. Take $n' \geq \max\left\{n, \frac{1}{r'}\right\}$. Then, for $n \geq n'$, $x_n \in \mathcal{B}_{(X,d)}(x, r_x)$ and $\mathcal{B}_{(Y,d')}\left(y, \frac{1}{n}\right) \subseteq \mathcal{B}_{(Y,d')}(y, r_y)$ and also $\mathcal{B}_{(X,d)}(x, r_x) \times \mathcal{B}_{(Y,d')}\left(y, \frac{1}{n}\right) \subseteq$ *graph* φ. . Therefore,

$$\forall n \geq n', \quad \text{there exists } y_n \in Y \text{ such that } y_n \in \mathcal{B}_{(Y,d')}\left(y, \frac{1}{n}\right) \text{ and } y_n \in \varphi(x_n),$$

as desired. ■

Proposition 65 *If $\varphi : X \rightrightarrows Y$ is non-empty valued and*

$$\text{for any } y \in Y, \ \ \varphi^{-1}(y) := \{x \in X : y \in \varphi(x)\} \text{ is open,}$$

then φ is lower semicontinuous.

Proof. Again, we are going to show that for any $x \in X$ and for every sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$, and for every $y \in \varphi(x)$, there exists a sequence $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $\forall\ n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ and $y_n \to y$. Since by assumption φ is not empty-valued, then we can take $y \in \varphi(x)$ and therefore $x \in \varphi^{-1}(y)$. Since $\varphi^{-1}(y)$ is open by assumption, then there exists $N \in \mathbb{N}$ such that for any $n > N$, $x_n \in \varphi^{-1}(y)$, i.e., for any $n > N$, $y \in \varphi(x_n)$. Then, for any $n > N$, take $y_n = y$, and the proof is concluded.

Definition 66 *A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, with X convex, is said to be*

(i) quasiconcave *iff for any $x, y \in X$ and $\lambda \in [0,1]$ one has*

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\};$$

(ii) semistrictly quasiconcave *iff for any $x, y \in X$ such that $f(x) \neq f(y)$, one has*

$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}, \quad \forall \lambda \in (0,1);$$

Definition 67 *Given a function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, we say that f is*

- *Locally NonSatiated if $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x' \in \mathcal{B}(x, \varepsilon)$ such that $f(x') > f(x)$;*
- *(Globally) NonSatiated if $\forall x \in X$ $\exists x' \in X$ such that $f(x') > f(x)$.*

Proposition 68 *If X is a convex metric space and $f : X \to \mathbb{R}$ is continuous, then*
f is semistrictly quasiconcave and NonSatiated if and only if f is quasiconcave and Locally NonSatiated.

Proposition 69 *(Proposition 2.38, page 50 in [22]) If a set-valued function $\varphi : X \rightrightarrows Y$ is lower semicontinuous if and only if $\mathrm{Cl}(\varphi)$ is.*

Proposition 70 *(Proposition 3, page 24 in [20]) If a set-valued function $\varphi : X \rightrightarrows Y$ is upper semicontinuous and compact valued, and $A \subseteq X$ is a compact set, then $\varphi(A)$ is compact.*

Proposition 71 *(Proposition 4, page 25 in [20]) For any $i \in \{1, ..., n\}$, let the the set-valued functions $\varphi_i : X \rightrightarrows Y$ be compact valued and upper semicontinuous at $x \in X$. Then, the set-valued function $\varphi : X \rightrightarrows Y^n$, $\varphi(x) = \times_{i=1}^n \varphi_i(x)$ is compact valued and upper semicontinuous at x.*

Proposition 72 *(Proposition 8, page 27 in [20]) For any $i \in \{1, ..., n\}$, let the the set-valued functions $\varphi_i : X \rightrightarrows Y$ be lower semicontinuous at $x \in X$. Then, the set-valued function $\varphi : X \rightrightarrows Y^n$, $\varphi(x) = \times_{i=1}^n \varphi_i(x)$ is lower semicontinuous at x.*

Proposition 73 *(Theorem 2.2.6, page 57 in [32]) If $C \subseteq \mathbb{R}^n$ is compact, then conv(C) is compact.*

Proposition 74 *(Lemma 1, page 33, in [20]) If a set-valued function φ of a metric space in $\mathbb{R}^n$ is non-empty valued, compact valued, convex valued, closed and lower semicontinuous. Then φ is upper semicontinuous.*

Proposition 75 *([20], bottom page 23) If $\varphi : X \rightrightarrows Y$ is a set valued function which is closed graph and if $\varphi(X)$ is contained in a compact set, then φ is upper semicontinuous.*

■

Proposition 76 *([10])If f is upper semicontinuous, quasiconcave and $\mathrm{Int}(\{x \in X : f(x) \geq f(\overline{x})\}) \neq \varnothing$, then*

$$\forall x \in X, (\{x \in X : f(x) > f(\overline{x})\}) = \mathrm{Int}(\{x \in X : f(x) \geq f(\overline{x})\}).$$

Proposition 77 *(Theorem 8.1, page 231 in [2]) If*
1. C is a nonempty, compact and convex subset of $X \subseteq \mathbb{R}^n$, and
2. $F : X \subseteq \mathbb{R}^l \rightrightarrows \mathbb{R}^l$ is nonempty valued, compact valued, convex valued and upper semicontinuous on C,
then there exists a solution $(\overline{x}, \overline{u})$ to the following Generalized Variational Inequality Problem.

$$\text{“Find } \overline{x} \in C \text{ and } \overline{u} \in F(\overline{x}) \text{ such that } \forall z \in C, \quad \langle \overline{u}, z - \overline{x} \rangle \geq 0\text{”.}$$

Theorem 78 *([31])If 1. C is a nonempty, compact and convex subset of $X \subseteq \mathbb{R}^l$,*
2. $K : X \rightrightarrows \mathbb{R}^l$ is nonempty, compact, convex valued, closed and lower semicontinuous on C,
2'. $K(C) \subseteq C$,
3. $F : X \rightrightarrows \mathbb{R}^l$ is nonempty, compact, convex valued and upper semicontinuous on C,
then the GQVI Problem (46) admits at least a solution.

6.2 Results to relate our work with the one available in the literature

Proposition 79 *1. $f : X \subseteq \mathbb{R}^C \to \mathbb{R}$ is quasiconcave if and only if for any $x \in X$, $P(x)$ is convex.*
2. If $f : X \subseteq \mathbb{R}^C \to \mathbb{R}$ is radially continuous on X with respect to X, then P satisfies Assumption (iii).
3. The opposite implication of the one presented in 2. is false

Proof. 1. Obvious.
2. We want to show that for any $x \in X$, $y \in P(x)$ and $z \in X \backslash \{y\}$, we have $[z, y) \cap P(x) \neq \varnothing$, i.e., in terms of the function f, for any $x \in X$, $y \in X$ such that $f(y) > f(x)$ and $z \in X \backslash \{y\}$, we have that there exists $w \in [z, y)$ such that $f(w) > f(x)$.
If $f(z) > f(x)$, then $z \in P(x)$. If $f(z) \leq f(x)$, then proceed as follows. Consider the segment $[z, y]$ and observe that by assumption $f(z) \leq f(x) < f(y)$. Define the function $g : [0, 1] \to \mathbb{R}$, $g(t) = f((1-t)z + ty)$ and observe that $g(0) = f(z) \leq f(x) < f(y) = g(1)$; moreover, since f is radially continuous, then g is continuous. Then from the intermediate value Theorem for continuous functions applied to g, we have that there exists $t_w \in (0, 1)$ such that $g(t_w) > f(x)$, i.e., there exists $w := (1 - t_w)z + t_w y$ such that $g(t_w) = f(w) > f(x)$. In other words, there exists $w \in (z, y)$ such that $f(w) > f(x)$.
3.Let the following function be given.[16]

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = \begin{cases} \frac{x_1 \cdot (x_2)^2}{(x_1)^2 + (x_2)^4} & \text{if} \quad x_1 \neq 0 \\ 0 & \text{if} \quad x_1 = 0. \end{cases}$$

f admits directional derivatives in *zero* in any direction and therefore it is radially continuous in *zero* (f is not continuous in *zero*). Now take $(y_1, y_2) \in \mathbb{R}^2$ such that $(y_1, y_2) \in P(0)$. i.e., such that $y_1 > 0$ and arbitrary $(z_1, z_2) \in \mathbb{R}^2 \backslash \{(y_1, y_2)\}$.Then taken $y(\lambda) := (1-\lambda)(z_1, z_2) + \lambda(y_1, y_2)$ with $\lambda \in (0, 1]$ and sufficiently close to zero, since f is continuous in (y_1, y_2), we have that $y(\lambda) \in P(0)$, as desired. ■

Proposition 80 *If $\overline{x}$ solves*

$$\text{(GVI-cone } (K, P)) \qquad \text{Find } \overline{x} \in K \text{ such that } \exists\, h \in N^{>}(\overline{x}) \backslash \{0\} \text{ such that } \forall x \in K, \quad \langle h, x - \overline{x} \rangle \geq 0, \tag{60}$$

*and either 1. Assumptions (i.2), (iii), (iv), (v), (vi) or 2.Assumptions (1.2**), (iii), (iv), (v), (vi) are satisfied, then $\overline{x}$ solves $M(K, P)$.*

Proof. The proof below is similar to the proof of Proposition 31.1.
1.
From Remark 25 and Assumption (iv), we know that $N^{>}(x)$ is a convex cone. Now assume our claim is false, i.e., while from (60), $\overline{x} \in K$, we have that $\overline{x} \notin M(K, P)$, i.e.,

$$P(\overline{x}) \cap K \neq \varnothing. \tag{61}$$

From (60), we have that

$$h \in N^{>}(\overline{x}) \backslash \{0\}. \tag{62}$$

Since $h \in N^{>}(\overline{x})$, we have that

$$\forall w \in P(\overline{x}), \quad \langle h, w - \overline{x} \rangle \leq 0. \tag{63}$$

By assumption (i.2), $K^{\perp} = \{0\}$. From (62) $h \neq 0$, then $h \notin K^{\perp}$, i.e., there exists $y \in K$ such that $\langle h, y - \overline{x} \rangle \neq 0$. From (60),

$$\exists y \in K \text{ such that } \langle h, y - \overline{x} \rangle > 0. \tag{64}$$

From (61), there exists $x \in X$ such that

$$x \in P(\overline{x}) \cap K. \tag{65}$$

Now observe that $y \neq x$. Indeed, if $y = x$, then from (64), we would have $\langle h, x - \overline{x} \rangle > 0$. Since from (65), we have $x \in P(\overline{x})$ and $h \in N^{>}(\overline{x})$, we also have $\langle h, x - \overline{x} \rangle \leq 0$, a contradiction.
Since, from (65), $x \in P(\overline{x})$ and $y \neq x$, we can use Assumption (iii), identifying $x \in X, y \in P(x), z \in X \backslash \{y\}$ there with $\overline{x} \in X, x \in P(\overline{x}), y \in K \backslash \{x\} \subseteq X \backslash \{x\}$ here, respectively to conclude that

$$[y, x) \cap P(\overline{x}) \neq \varnothing. \tag{66}$$

For any $\lambda \in (0, 1]$, define

$$x(\lambda) = (1-\lambda)x + \lambda y.$$

[16] The example below is a textbook example of the fact that the existence of directional derivative in a point in any direction does not imply continuity in that point −see [3], page 345.

Then

$$\langle h, x(\lambda) - \overline{x} \rangle = \langle h,\ (1-\lambda)x + \lambda y - \overline{x} + \lambda\overline{x} - \lambda\overline{x} \rangle = \lambda \langle h, y - \overline{x} \rangle + (1-\lambda) \langle h, x - \overline{x} \rangle > 0,$$

where the strict inequality follows from (64), (60) and the fact that $\lambda \in (0,1]$. Therefore, from (63), $\forall \lambda \in (0,1]$, $x(\lambda) \notin P(\overline{x})$, i.e.,

$$[y, x) \cap P(\overline{x}) = \varnothing,$$

contradicting (66).

2.

Suppose otherwise, i.e., since $\overline{x} \in K$, then

$$\text{there exists } x' \in K \text{ such that } x' \in P(\overline{x}). \tag{67}$$

Then, from Assumption (60), we have

$$h \in N^{>}(\overline{x}) \setminus \{0\} \quad \text{and} \quad \langle h, x' - \overline{x} \rangle \geq 0. \tag{68}$$

Since $h \in N^{>}(\overline{x})$ and since $x' \in P(\overline{x})$, we have $\langle h, x' - \overline{x} \rangle \leq 0$, and therefore

$$\langle h, x' - \overline{x} \rangle = 0. \tag{69}$$

From Assumption (i.2**) and from (67), we have that there exists $\varepsilon > 0$ such that $\mathcal{B}(x', \varepsilon) \subseteq X$ and therefore, since $h \neq 0$, there exists $\alpha > 0$ such that $x' + \alpha h \in X$. Since, from (67), $x' \in P(\overline{x})$ and $\overline{x} \neq x'$, we can use Assumption (iii) to conclude that $[x' + \alpha h, x') \cap P(\overline{x}) \neq \varnothing$, i.e., there exists $\alpha^* \in (0, \alpha)$ such that $x' + \alpha^* h \in P(\overline{x})$. Then, since $h \in N^{>}(\overline{x})$, $\langle\ h, (x' + \alpha^* h) - \overline{x} \rangle \leq 0$. Finally,

$$0 \geq \langle\ h, (x' + \alpha^* h) - \overline{x} \rangle = \langle\ h, x' - \overline{x} \rangle + \alpha^* \|h\|^2 \overset{(69)}{=} \alpha^* \|h\|^2 > 0,$$

the desired contradiction. ■

Proposition 81 *If f is upper semicontinuous at x and x is not a global maximum for f, then P is lower semicontinuous at x.*

Proof. Recall that $f : X \to \mathbb{R}$ is upper semicontinuous at $\overline{x}$ if

$$\forall \lambda > f(\overline{x}), \quad \exists \delta > 0 \quad \text{such that } x \in B(\overline{x}, \delta) \Rightarrow \lambda > f(x).$$

We want to show that P is lower semicontinuous at $x \in X$, i.e., $P(x) \neq \varnothing$ and for every sequence $(x_n)_{n \in \mathbb{N}} \in X^\infty$ such that $x_n \to x$, and for every $y \in P(x)$,there exists a sequence $(y_n)_{n \in \mathbb{N}} \in X^\infty$ such that $\forall\ n \in \mathbb{N}$, $y_n \in P(x_n)$ and $y_n \to y$. Nonemptyness follows from the fact x is not a global maximum for f. Since $y \in P(x)$, then $f(y) > f(x)$. Since f is upper semicontinuous at $\overline{x}$, then (identified $f(y)$ with λ), we have that $\exists \delta > 0$ such that for any $z \in B(x, \delta)$, we have $f(y) > f(z)$. Since $x_n \to x$, then there exists $N \in \mathbb{N}$ such that $\forall n > N$, $x_n \in B(x, \delta)$ and therefore $f(y) > f(x_n)$, i.e., for any $n > N$, $y \in P(x_n)$. Taking $y_n = y$ for any $n > N$ concludes the proof. ■

Proposition 82 *1. If $\succ$ is upper semicontinuous, then P is lower semicontinuous;*
2. The opposite implication is false.

Proof. Recall that

$$\langle \succ \text{ is upper semicontinuous} \rangle :=$$

$$\left\langle\ (x_n)_{n \in \mathbb{N}} \subseteq X^\infty, x_n \longrightarrow x, \quad y \succ x \right\rangle \Rightarrow \langle\ \exists \nu \in \mathbb{N} \text{ such that } \forall m > \nu, \quad y \succ x_m \rangle$$

$$\langle\ P \text{ is lower semicontinuous} \rangle :=$$

$$\left\langle\ (x_n)_{n \in \mathbb{N}} \subseteq X^\infty, x_n \longrightarrow x, \quad y \succ x \right\rangle \Rightarrow \left\langle\ \exists (y_n)_{n \in \mathbb{N}} \in X^\infty \text{ such that i. for any } n \in \mathbb{N},\ y_n \succ x_n, \text{ and ii. } y_n \longrightarrow y \right\rangle$$

1.

Take $y_{\nu+n} = y$ for any $n \in \mathbb{N}$ and consider the sequence $(y_{\nu+n})_{n \in \mathbb{N}}$. Then, by assumption, since $\nu + n > n$, we have $y_{\nu+n} \succ x_{\nu+n}$ and $y_n \longrightarrow y$.

2.

Take $P : \mathbb{R}_+ \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}$,

$$P(x) = \begin{cases} \{2\} & \text{if} \quad x \in [0,1], \\ [1,3] \setminus \{2\} & \text{if} \quad x > 1. \end{cases}$$

P is lower semicontinuous

Indeed the only point to be checked is $x = 1$ for which $P(x) = \{2\}$. If $x_n \longrightarrow 1$ with $x_n \leq 1$, take $y_n = 2$ for any $n \in \mathbb{N}$; If $x_n \longrightarrow 1$ with $x_n \geq 1$, take $y_n = -e^{-x_n+1} + 3$ for any $n \in \mathbb{N}$.

$\succ$ is not upper semicontinuous. Take $x_n = 1 + \frac{1}{n}$, $y = 2$. Then, for any $n \in \mathbb{N}$, $2 \notin P(x_n)$. ■

Proposition 83 *Assume that $f : X \subseteq \mathbb{R}^C \to \mathbb{R}$ has no global maximum.*
1. a. f continuous $\Rightarrow$ (iii) and b. not vice versa;
2. a. f continuous $\Rightarrow$ (ii) and b. not vice versa.
3. f continuous $\Rightarrow$(vii) $\Rightarrow$ (vii).*
4a. f continuous and semistrictly quasi concave $\Rightarrow$ f quasi concave and b. not vice versa.

Proof. 1a.
Observe that f continuous $\Rightarrow$ $P(x)$ open $\Rightarrow$ (iii), where the first implication is obvious and the second one is verified below.
If P is open valued and $y \in P(x)$, then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq P(x)$. Then for any $z \in X \backslash \{y\}$, define $h = z - y \in X \backslash \{0\}$and $\overline{\alpha} = \frac{\varepsilon}{2\|h\|}$. Then, for any $\alpha \in [0, \overline{\alpha}]$, we have $\|y - (y + \alpha h)\| = \|ah\| = \alpha \|h\| < \frac{\varepsilon}{2\|h\|} \|h\| < \varepsilon$, and therefore $y + \alpha h \in B(y, \varepsilon) \subseteq P(x)$.
1b.
From Proposition 79, we have what follows

$$f \text{ continuous} \quad \begin{array}{c}\Rightarrow \\ \\ \not\Leftarrow\end{array} \quad f \text{ radially continuous} \quad \begin{array}{c}\Rightarrow \\ \\ \not\Leftarrow\end{array} \quad \text{Assumption (iii).} \tag{70}$$

If our Claim were false, we would Assumption (iii) $\Rightarrow$ f continuous and therefore f radially continuous, contradicting (70).
2a. Since f continuous implies f upper semicontinuous, it is enough to show that f upper semicontinuous implies P is lower semicontinuous, which is done in Proposition 81.
2b. Below we present an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous and whose associated P is lower semicontinuous. Define

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 1 \\ 2 & \text{if} \quad x = 1 \\ 2 + x & \text{if} \quad 1 < x \end{cases}$$

Then

$$P(x) = \begin{cases} [1, +\infty) & \text{if} \quad x \in (-\infty, 1) \\ (1, +\infty) & \text{if} \quad x = 1 \\ (x, +\infty) & \text{if} \quad x \in (1, +\infty) \end{cases}$$

is lower semicontinuous, the only points need a simple checking being $x = 1$.
3. obvious.
4a. It is the content of Proposition 4.16, page 154, in [19].
4b. Obvious. ■

Proposition 84 *([8] and [33]) Given $x \in X$, if $P^{-1}(x)$ is open and $P(x) \neq \varnothing$, then P is lower semicontinuous at x. The opposite implication is not true.*

Proof. We want to use the following characterization of lower semicontinuity. A set valued function $\varphi : X \to\to Y$ is lower semicontinuous if for every open set V in Y, $\{x \in X : \varphi(x) \cap V \neq \varnothing\}$ is open in X `(see for example, my math 2 notes)`. Observe that

$$\{x \in X : \varphi(x) \cap V \neq \varnothing\} = \cup_{y \in V} \varphi^{-1}(y),$$

as verified below.
$z \in \{x \in X : \varphi(x) \cap V \neq \varnothing\} \Leftrightarrow$ there exists $y \in \varphi(z) \cap V$.
$z \in \cup_{y \in V} \varphi^{-1}(y) \Leftrightarrow$ there exists $y \in V$ such that $z \in \varphi^{-1}(y) := \{x' \in X : y \in \varphi(x')\} \Leftrightarrow$ there exists $y \in V$ such that $y \in \varphi(z)$.
For the proof about the opposite implication, see Remark 4.1, page 237, in [33]. ■

6.3 The counterexample in [4]

We proceed as follows.
1. Introduce the example presented by Aussel in [4] using our framework;
2. Show that in that example,
i. all the assumptions of theorem 3.b. in [27] are satisfied: K is nonempty, convex and closed; $U(x)$ is open (and indeed preferences can be represented by a continuous utility function);
ii. $\overline{x}$ solves a GVI but it is not a maximal element (because there are no maximal elements).

iii. Observe that example does not show that our Proposition 31 is false, because the example is such that Assumptions (i.2) and (i.2**) are violated.
1.
Take $X = \mathbb{R}_+ \times \mathbb{R}$, $K = \{0\} \times \mathbb{R}$ and $u : X \to \mathbb{R}$, $(x_1, x_2) \mapsto x_2$. The maximization problem we want to analyze is the following one.

$$\max_{(x_1,x_2)\in\mathbb{R}_+\times\mathbb{R}} \quad x_2 \quad s.t. \quad x \in \{0\} \times \mathbb{R}.$$

2.
(i.)
K is nonempty, convex and closed; $P(x)$ is open, and indeed preferences are represented by the continuous utility function $u : X \longrightarrow \mathbb{R}$, $u(x_1, x_2) = x_2$.
(ii.)
We want to show that it is false that

$$\overline{x} \in K \text{ solution to (GVI) } \exists h \in G(\overline{x}) : \langle h, x - \overline{x}\rangle \geq 0, \forall x \in K \quad \Rightarrow \overline{x} \in \arg\max_{x\in K} u(x).$$

Observe that, obviously, $\arg\max_{x\in K} u(x) = \varnothing$.
We are going to show that the set of solutions to GVI is K.
First of all, we have to describe G. Indeed, we are going to show that $G : \mathbb{R}_+ \times \mathbb{R} \rightrightarrows \mathbb{R}^2$,

$$G(x_1, x_2) = \begin{cases} \{(0, -1)\} & \text{if} \quad x_1 > 0, \\ \\ \text{conv}\left(-\mathbb{R}^2_+ \cap S(0, 1)\right) & \text{if} \quad x_1 = 0. \end{cases}$$

That simple result from a geometrical viewpoint does require some work.
First of all, observe that for any $\overline{x} \in X$, $y \in P(\overline{x}) \Leftrightarrow y_1 \geq 0$ and $y_2 \geq \overline{x}_2$. Since $\arg\max_{x\in\mathbb{R}_+\times\mathbb{R}} u(x_1, x_2) = \varnothing$, we do have $G(\overline{x})$ =conv($N^>(\overline{x}) \cap S(0, 1)$) and we have to compute

$$\begin{aligned} &N^>(\overline{x}) := \{h \in \mathbb{R}^n : \forall y \in U^>(\overline{x}), \ \langle h, y - \overline{x}\rangle \leq 0\} = \\ &= \left\{(h_1, h_2) \in \mathbb{R}^2 : \forall y_1 \geq 0, \forall y_2 > \overline{x}_2, (h_1, h_2)(y_1 - \overline{x}_1, y_2 - \overline{x}_2) \leq 0\right\}. \end{aligned} \tag{71}$$

Claim.

$$N^>(\overline{x}) = \begin{cases} \{0\} \times (-\mathbb{R}_+) & \text{if} \quad \overline{x}_1 > 0, \\ \\ -\mathbb{R}^2_+ & \text{if} \quad \overline{x}_1 = 0. \end{cases}$$

Proof of the Claim.
$\boxed{\text{Case A. } \overline{x}_1 > 0.}$
Consider the main inequality in the definition of $N^>(\overline{x})$:

$$h_1 \cdot (y_1 - \overline{x}_1) + h_2 \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0. \tag{72}$$

<u>Subclaim.</u> $h_1 = 0$.
<u>Proof of the Subclaim.</u> Suppose otherwise. Consistently with the description of $N^>(\overline{x})$ presented in (71), the idea of what follows is to find $(y_1, y_2) \in \mathbb{R}^2$ such that $y_1 \geq 0$ and $y_2 > \overline{x}_2$ which violates inequality (72).
Case 1. $h_1 > 0$ and $h_2 \geq 0$.

$$\overset{(>0)}{h_1} \cdot \overset{(\gtreqless 0)}{(y_1 - \overline{x}_1)} + \overset{(\geq 0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0.$$

If $y_1 > \overline{x}_1$, then (72) is violated.
Case 2. $h_1 > 0$ and $h_2 \leq 0$.

$$\overset{(>0)}{h_1} \cdot \overset{(\gtreqless 0)}{(y_1 - \overline{x}_1)} + \overset{(\leq 0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0.$$

If $y_1 > \overline{x}_1$ and sufficiently large, then (72) is violated.
Case 3. $h_1 < 0$ and $h_2 \geq 0$.

$$\overset{(<0)}{h_1} \cdot \overset{(\gtreqless 0)}{(y_1 - \overline{x}_1)} + \overset{(\geq 0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0.$$

If $0 \leq y_1 < \overline{x}_1$, then (72) is violated.
Case 4. $h_1 < 0$ and $h_2 \leq 0$.

$$\overset{(<0)}{h_1} \cdot \overset{(\gtreqless 0)}{(y_1 - \overline{x}_1)} + \overset{(\leq 0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0.$$

Take $y_1 = \overline{x}_1 - \frac{1}{m}$ and $y_2 = \overline{x}_2 + \frac{1}{n}$ with $n, m \in \mathbb{N}$. Observe that since $\overline{x}_1 > 0$, by assumption of the case, there exists $m_0 \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ and $m > m_0$, $y_1 = \overline{x}_1 - \frac{1}{m} > 0$. Then we want to show the following statement is false

$$\exists h_1 < 0, h_2 \leq 0 \text{ such that } \forall n \in \mathbb{N} \text{ and } \forall m \in \mathbb{N} \text{ and } m > m_0, \text{ we have } \overset{(<0)}{h_1} \cdot \left(-\frac{1}{m}\right) + \overset{(\leq 0)}{h_2} \cdot \frac{1}{n} \leq 0,$$

i.e.,

$$\forall h_1 < 0, \forall h_2 \leq 0, \exists n, m \in \mathbb{N} \text{ with } m > m_0 \text{ such that } \overset{(<0)}{h_1} \cdot \left(-\frac{1}{m}\right) + \overset{(\leq 0)}{h_2} \cdot \frac{1}{n} > 0,$$

i.e.,

$$\forall h_1 < 0, \forall h_2 \leq 0, \exists n, m \in \mathbb{N} \text{ and } m > m_0 \text{ such that } \frac{\overset{(\leq 0)}{h_2}}{\overset{(<0)}{h_1}} < \frac{n}{m},$$

which is true for n sufficiently large.
End of the proof of the Subclaim.
We are now left with showing that $h_2 \leq 0$. From the above Subclaim, we have that (72) becomes

$$h_2 \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0,$$

which is certainly true for $h_2 \leq 0$, as desired.
$\boxed{\text{Case B. } \overline{x}_1 = 0.}$
We want to show that

$$N^{>}(\overline{x}) := \left\{(h_1, h_2) \in \mathbb{R}^2 : \forall y_1 \geq 0, \forall y_2 > \overline{x}_2, h y_1 + h_2 (y_2 - \overline{x}_2) \leq 0\right\} = -\mathbb{R}^2_+.$$

[⊇]
Indeed,

$$\overset{(\leq 0)}{h_1} \cdot \overset{(\geq 0)}{y_1} + \overset{(\leq 0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0$$

is certainly verified.
[⊆]
Suppose otherwise, i.e., either $h_1 > 0$ or $h_2 > 0$.
If $h_1 > 0$, then

$$\overset{(>0)}{h_1} \cdot \overset{(\geq 0)}{y_1} + \overset{(\gtrless 0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0$$

which is violated for large enough y_1.

If $h_2 > 0$, then

$$\overset{(\gtrless 0)}{h_1} \cdot \overset{(\geq 0)}{y_1} + \overset{(>0)}{h_2} \cdot \overset{(>0)}{(y_2 - \overline{x}_2)} \leq 0$$

which is violated for $y_1 = 0$.
End of the proof of the Claim.
Summarizing, we have that $G : \mathbb{R}_+ \times \mathbb{R} \to\to \mathbb{R}^2$,

$$G(\overline{x}_1, \overline{x}_2) = \operatorname{conv}\left(N^{>}(\overline{x}) \cap S(0,1)\right) = \begin{cases} \{(0,-1)\} & \text{if } \overline{x}_1 \neq 0, \\ \operatorname{conv}\left(-\mathbb{R}^2_+ \cap S(0,1)\right) & \text{if } \overline{x}_1 = 0. \end{cases}$$

We can now rewrite problem GVI as follows.

$$\text{Find } (\overline{x}_1, \overline{x}_2) \in \{0\} \times \mathbb{R} \text{ such that } \exists\ (h_1, h_2) \in G(\overline{x}_1, \overline{x}_2) \text{ such that}$$

$$\forall (x_1, x_2) \in \{0\} \times \mathbb{R}, \quad \langle (h_1, h_2), (x_1, x_2) - (\overline{x}_1, \overline{x}_2) \rangle \geq 0,$$

i.e.,

$$\text{Find } \overline{x}_2 \in \mathbb{R} \text{ such that } \exists\ (h_1, h_2) \in G(0, \overline{x}_2) = \operatorname{conv}\left(-\mathbb{R}^2_+ \cap S(0,1)\right) \text{ such that}$$

$$\forall x_2 \in \mathbb{R}, \quad \langle (h_1, h_2), (0, x_2) - (0, \overline{x}_2) \rangle = h_2 (x_2 - \overline{x}_2) \geq 0,$$

i.e.,

$$\text{Find } \overline{x}_2 \in \mathbb{R} \text{ such that } \exists \ (h_1, h_2) \in \text{conv}\left(-\mathbb{R}^2_+ \cap S(0,1)\right) \text{ such that } \forall x_2 \in \mathbb{R}, \quad h_2(x_2 - \overline{x}_2) \geq 0.$$

To have $\forall x_2 \in \mathbb{R}, \quad h_2(x_2 - \overline{x}_2) \geq 0$, it has to be $h_2 = 0$ and then since $(h_1, h_2) \in \text{conv}\left(-\mathbb{R}^2_+ \cap S(0,1)\right)$, it must be $h_1 = -1$. Then, the set of solutions to (GVI) above is $\{0\} \times \mathbb{R}$, as desired.*
iii.
Recall that $K = \{0\} \times \mathbb{R}$.
Assumption (i.2) is violated.
Indeed,

$$K^{\perp} = \mathbb{R} \times \{0\},$$

i.e., $K^{\perp}$ is the horizontal axis, as verified below.
We want to show that

$$K^{\perp} := \left\{z \in \mathbb{R}^C : \forall x, y \in K, \langle z, x - y\rangle = 0\right\} = \mathbb{R} \times \{0\}.$$

First proof.
[$\supseteq$]. We want to show that for any $\zeta \in \mathbb{R}$, $(\zeta, 0) \in K^{\perp}$, i.e., we have for any $(0,\xi), (0,\theta) \in K$, we have $\langle(\zeta, 0), (0, \xi) - (0, \theta)\rangle = 0$, which is obvious.
[$\subseteq$]. We want to show that if $(z_1, z_2) \in K^{\perp}$, then $z_2 = 0$. Indeed,

$$(z_1, z_2) \in K^{\perp} \Rightarrow \forall \xi, \theta \in \mathbb{R}, \quad (z_1, z_2)(0, \xi - \theta) = 0.$$

Chosen $\xi = 1$ and $\theta = 0$, we have $0 = (z_1, z_2)(0, 1) = z_2$, as desired.
Second proof.
Observe that $K^{\perp} = \{0\}$ if and only if aff $(K) = \mathbb{R}^2$; but

$$\text{aff}(K) := \text{aff } (\{0\} \times \mathbb{R}) \subseteq \text{ span}(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$$

Assumption (i.2**) is violated.
It is clearly false that $K \subseteq \text{Int}_{\mathbb{R}^C}(X)$.

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