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Large and uncertain expectations heterogeneity: equilibrium stability from a policy maker standpoint.

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Abstract

We study the dynamic behavior of heterogeneous markets with many agents' types. In particular this paper aims at studying the effects of a change in the number n of agents, who are possibly different in terms of the rule they employ to forecast, on the long run value of a relevant state variable. On the one hand we show that a heterogeneous-agents model cannot be by and large traced back to an equivalent average-representative-agent model thus having a negative impact on the possibility of easily reducing large and complex models to simpler and analytically tractable ones. On the other hand, under fairly general conditions, we characterize a class of models in which the probability of convergence to the steady state becomes either one or zero as n grows. This fact has positive implications from the point of view of a policy maker willing to take action toward the goal of stabilizing the economy.

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1 Introduction

In the economic literature, heterogeneity in behaviour or capacity to forecast used to be frequently overlooked by claiming (implicitly or explicitly) a sort of equivalence between the full model with heterogeneous agents and a corresponding model with an *average representative agent*. Over the last twenty years, however, this approach has raised a lot of criticism in many contexts: for example [17, 45] within macroeconomic modelling, [30] in the field of choice under risk, [25] in the context of measures of social welfare, to name only a few. In particular, from the point of view of *out-of-equilibrium* dynamics, it is now widely accepted that such a simplified (average) model may generate different dynamic outcomes with respect to those attained when heterogeneous agents are explicitly introduced. On the contrary, when the focus of the analysis is on local dynamics, the representative agent assumption is still considered to be a good compromise granting analytical tractability at the cost of minor shortcomings. This point of view has been epitomized by Grandmont who, studying local stability conditions of the self-fulfilling equilibrium in an adaptive learning economy in [29], reckoned that

> "... the methods described below [regarding a representative agent economy] can nevertheless be made to bear upon the case of heterogeneous beliefs. The forecast ${}_t x^e_{t+1}$ can then be reinterpreted as an average forecast, each individual forecast being weighted by its relative local contribution to the dynamics of the system ...".

This equivalence however only holds under suitable assumptions. For example, [20] shows that local stability conditions under heterogeneity differ from those with a representative agent even in the standard cobweb model with adaptive expectations.[1]

There is indeed a large body of literature showing evidence of heterogeneity in expectations, both in empirical contexts, e.g. [15], and in the lab, e.g. [35]. From the point of view of modelling markets with heterogeneous agents and expectations it has been pointed out that there is a risk of falling in the "wilderness" of bounded rationality, characterized by an excess of degrees of freedom and parameters. Brock et al. [13] address the issue of dimensionality reduction in a fairly general theoretical framework with many types and develop an analytical

[1] The problem of variables/data aggregation to the purpose of simplification is common to various economics fields, especially Macroeconomics and Econometrics. For example the issue of aggregation over heterogeneous individuals when their composition is dynamically changing has been addressed in [28] and [46]. The representativeness of the representative agent, both in economic theory and econometrics has been extensively discussed in [37] and [31]. Finally, the relevance of using agent-based computational economics models as a tool to address the aggregation problem and the analogy principle has been discussed, among others in [27]. [16] is recent survey on this subject.

result, namely the notion of *Large Type Limit*, based on the idea of substituting the (large) array of stochastic parameters describing the populations of agents with the (small) parameter vector defining their joint probability distribution, hence reducing the model complexity. In particular individual types are sampled at the beginning of the market, heterogeneity is aggregated linearly in the function defining the dynamics of the state variable, and agents are allowed to switch over time among types (see [11, 12]): in this setup, the Large Type Limit entails replacing sample moments with population moments, which for particular distributions of characteristics yield closed form expressions. The result in [13] has been applied in several papers[2] among which [2], [1] and [40], in order to analyze monetary policy issues, taking advantage of the assumption that the forecasting rules are actually constants.

Building on [13], Diks and van der Weide [24] assume the distribution of beliefs among agents is updated using a *Continuous Choice Model*,[3] which leads to price dynamics in which the beliefs' distribution evolves together with realized prices. They call this setup *Continuous Belief Systems*, which give rise to random dynamical systems, containing deterministic dynamics as a special case (and therefore generalizing the Large Type Limit setup). The issue of reducing complex agent-based economic models to analytically tractable small-scale ones has also been addressed, among others in [26], and more recently in [42].

This paper studies a class of models in which there is a significant interplay between uncertainty about behavioural parameters of agents, which pertains to an observer (e.g. a policy maker), and how heterogeneous such agents are. It appears natural to expect that, as more heterogeneity affects the system, the task of predicting its fate (for example computing the probability that the system eventually converges to a steady state) becomes ever more difficult. However, a mildly surprising asymptotic result contradicts such intuition in a variety of contexts: it may in fact happen that when the amount of heterogeneity goes to infinity (in a sense to be specified) the probability of convergence becomes either one or zero depending on the value of crucial structural parameters of the economy. We describe this phenomenon as *polarization.* The amount of the heterogeneity and its possible variations hence play a critical role in shaping the range of possible long-run outcomes of the model. Various implications seem to be the result of polarization. Policy-wise, knowing that such an effect is in place and that heterogeneity is relatively high may encourage policy makers to take action towards stabilizing the system. Similar types of implications have been reached in contexts close to the present one e.g. [14] and [21].

The present work rests on a number of simplifying assumptions which need to be briefly discussed in order to lay out the perimeter of this paper. One aspect is the fact that the possibility of agents switching between alternative forecast rules is precluded. In other words we are assuming that agents stick to forecasting rules, which is certainly oversimplifying. Learning to forecast lab experiments have in fact shown that individuals use a variety of simple heuristics (see [35], [4]) but the evidence in favor of evolutionary switching among rules does not appear to be overwhelming.[4] However our main reason to relinquish this mechanism here, lies in the fact that doing so allows us "... to derive an analytically tractable system with a unique equilibrium, which is desirable from a policy maker view point ...".[5] Our perspective in this paper is indeed that of an observer or a policy maker whose main interest is that of deriving conditions favouring long run stability of the economy. On one hand therefore our results here are of a more limited nature with respect to, e.g. [13], where large expectations heterogeneity is coupled with endogenous switching between forecast rules. At the same time however, the core of this paper, namely Section 3, explores a model in which the explicit dependence of expectations at time t on previous expectations is outside the range of a key assumption in [13].[6] Our approach to expectations, which are assumed to have some degree of inertia, is supported by evidence that in complex environments observable behaviour is, loosely speaking, adaptive (see e.g. [32, 35]).

The paper is organized as follows. Section 2 introduces the baseline model for a scalar state variable and in particular delineates how it is stochastic and shows the conditions for local stability and polarization (i.e. what happens to probability of stability in the long run when heterogeneity goes to infinity). The model is generalized in Subsection 2.1 to allow for vector state variables and vector stochastic parameters. Section 3 adds inertia in expectations and studies how results change (given the explosion in the dimension of the random matrix whose spectral radius governs stability) as a consequence. Various generalizations as to the types of laws of motion

[2] [13] itself applies the Large Type Limit to a generalization of the asset pricing model introduced in [12], assuming agents who are choosing beliefs from a class of linear auto-regressive functions with several lagged variables.

[3] A generalization of the *Discrete Choice Model* used in [11]. See [38] for more details.

[4] Several authors have specifically estimated nonlinear switching mechanisms. For example, [10], [22], [41] and [7] do so using S&P 500 data, various commodities markets data, US inflation rate data and gold market data respectively, finding some evidence of switching.

[5] Quoted from [40]. The need for simplifying assumptions for the sake of analytical tractability is common in the literature. For example, while [13] shows that an equivalence exists between the dynamics of the large heterogeneous economy and the simpler corresponding Large Type Limit, it does not automatically imply the simpler limit model being analytically tractable. Indeed, to get a manageable model, [1], [2] and [40], assume that the forecasting rules are constants, matching in spirit our assumption of constant fractions.

[6] We are referring to assumption B3 in [13] in particular.

permitted and examples are presented in Subsection 3.1. Some discussion and directions in which to extend and further develop the present work are indicated in Section 4. Concluding remarks are offered in Section 5. The Appendix gathers all the proofs.

2 The baseline model

The general model we have in mind is one in which the dynamic evolution of a state variable of interest depends on two sets of elements: past values of the st ate variable itself and past and present values of another variable, typically (but not necessarily) describing expectations on the state (see, for example, [23] where the same approach is adopted). We'll move towards such general form starting from a relatively simple model and building up its complexity in various steps. In what follows we will refer to the x variable(s) as the state variable and to the y variable(s) as to expectations.

Consider the (random) discrete-time dynamical system

$$\begin{cases} x_{t+1} = f(y_{t+1}) \\ y_{t+1} = g(x_t, \alpha) \end{cases} \tag{1}$$

where $f \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})$ and $\alpha : \Omega \to A \subset \mathbb{R}$ a random variable, with mean $\mathbb{E}[\alpha]$ and variance $\mathbb{V}(\alpha)$ on a given probability space $(\Omega, \mathbb{F}, \mathbb{P})$. System (1) describes a single-agent model where y is the expectation on the state x, the function f is the law of motion whereas g represents the way expectations are computed, depending on the random parameter α. Replacing y_{t+1} in the first equation of (1) gives a difference equation which completely describes the system.[7] Assume $(x^*, y^*) \in \mathbb{R}^2$ is the unique steady state irrespective of α, i.e., $x^* = f(y^*)$ and $y^* = g(x^*, \alpha)$ for all $\alpha \in A$. In equilibrium, stability for the deterministic skeleton of (1) (i.e. for the degenerate case $\mathbb{V}(\alpha) = 0$) obtains if:

$$-1 < f_y(y^*) g_x(x^*, \alpha) < 1 \tag{2}$$

where the subscripts denote partial derivatives. In what follows, we will assume $f_y(y^*) \neq 0$ to avoid trivial cases. Generally, (2) results in a condition on α.

In general, the uncertainty on α reflects the difficulties in its estimation. The stability of (1) will depend on the particular realization of α. Let the set

$$S_1 = \{\omega \in \Omega : f_y(y^*) g_x(x^*, \alpha(\omega)) \in (-1, 1)\} \subset \Omega, \tag{3}$$

so the probability of obtaining a stable system can be computed as $\mathbb{P}(S_1)$.

We ask the following question. What happens if there are heterogeneous agents in the economy whose characterizing parameters must be estimated? To this end let $\{\alpha_i\}_{i=1}^n$ a discrete stochastic process on $(\Omega, \mathbb{F}, \mathbb{P})$. Agent heterogeneity lies in the fact that they use different prediction rules indexed by α_i. For any given n, we consider

$$y^i_{t+1} = g(x_t, \alpha_i), \quad i = 1, \dots, n \tag{4}$$

ignoring the α_i for $i > n$. Assume that heterogeneity is *aggregated* through linear combination:

$$y_{t+1} = \sum_{i=1}^{n} \phi_i \, y^i_{t+1}, \tag{5}$$

where the family of weights $\{\phi_i\}_{i=1}^n$ satisfies $0 \leq \phi_i \leq 1$, $\sum_{i=1}^n \phi_i = 1$ for all $n \in \mathbb{N}$ and $\phi_i = 0$ for $i > n$.[8] In other words, for any fixed n we consider as given the random variables $(\alpha_1, \dots, \alpha_n)$ and the weights $(\phi_1, \dots, \phi_n)$ summing up to 1. Substituting (4) in (5) and then in turn in (1), we obtain the dynamical system

$$\begin{cases} x_{t+1} = f(y_{t+1}) \\ y_{t+1} = \sum_{i=1}^n \phi_i \, g(x_t, \alpha_i) \end{cases} \tag{6}$$

The unique steady state of (6) is again $(x^*, y^*) \in \mathbb{R}^2$ and the probability of stability of (6) becomes $\mathbb{P}(S_n)$, where

$$S_n = \{\omega \in \Omega : f_y(y^*) \sum_{i=1}^{n} \phi_i \, g_x(x^*, \alpha_i(\omega)) \in (-1, 1)\}. \tag{7}$$

[7] Indeed, the y variable could be dropped altogether in this Section. However, in order to maintain consistency in the notation with the next Section (where such reduction is not possible) and to understand in what sense the model is progressively enriched, it is more suitable to keep the variable y, which, by the way, also has a relevant semantic interpretation (it represents expectations).

[8] Observe that the weights are not in general independent of n: so they should really be thought of as $\phi_i(n)$. We have relinquished such heavier notation for simplicity.

Remark 1 *The condition for stability of* (6) *and its probability* $\mathbb{P}(S_n)$ *remain the same under a different aggregation policy, such as* $x_{t+1} = \sum_{i=1}^n \phi_i f(y^i_{t+1})$ *instead of* (5)*. Therefore aggregation can be done as a linear combination of the law of motion* f *instead of mixing the variables* y^i*, without compromising the results.*

We are interested in the study of $\mathbb{P}(S_n)$, especially when n grows and tends to infinity. Beside explicit computations when f, g, ϕ_i and the exact distribution of the α_i are known, we wish to give general results concerning the asymptotic behavior of $\mathbb{P}(S_n)$. Let $C := \{\omega \in \Omega : \lim_n \sum_{i=1}^n \phi_i g_x(x^*, \alpha_i)(\omega) \text{ exists and is finite}\}$.

Proposition 2 *Assume the random variables* α_i *are independent. Then:*

(i) $\mathbb{P}(C) = 0$ *or* 1.

(ii) If both $\lim_n \sum_{i=1}^n \mathbb{E}[\phi_i g_x(x^*, \alpha_i)]$ *and* $\lim_n \sum_{i=1}^n \mathbb{V}(\phi_i g_x(x^*, \alpha_i))$ *converge, then* $\mathbb{P}(C) = 1$.

(iii) If $\lim_n \sum_{i=1}^n \phi_i^2 = 0$ *and* α_i *are identically distributed, then*

$$\lim_n \mathbb{P}(S_n) = \begin{cases} 1 & \text{if } f_y(y^*)\mathbb{E}[g_x(x^*, \alpha_1)] \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Proposition 2, *(iii)*, gives a precise result about the chances of obtaining a stable steady state for (6).[9] However, the polarization exhibited in (8) implies restrictions on the weights ϕ_i and the α_i. In particular, the assumption on ϕ_i, which for instance holds if $\phi_i = \frac{1}{n}$, demands that in the limit with a huge number of agents, each of them has a negligible impact over the aggregation of the expectations. Besides, α_i identically and independently distributed is also rather binding. Relaxing these hypotheses causes two main problems. First, we could have $\mathbb{P}(C) = 0$ (see Example 3). Second, we can have $\mathbb{P}(C) = 1$ but the almost sure limit $\lim_n \sum_{i=1}^n \phi_i g_x(x^*, \alpha_i)$ is a non-degenerate random variable: in fact, our limit can be any infinitely divisible random variable[10]. In such cases, the polarization shown in (8) is no longer granted (see Example 4 and Example 6).

Example 3 *We show that* $\mathbb{P}(C)$ *can be zero. For instance, this happens when the series* $\sum_i \phi_i g_x(x^*, \alpha_i)$ *diverges to infinity almost surely. If* $\phi_i = \frac{1}{n}$ $\forall i \le n$, $g_x(x^*, \alpha) = \alpha$ *and*

$$\alpha_i = \begin{cases} i & \text{with probability } \frac{1}{2} \\ i^2 & \text{with probability } \frac{1}{2} \end{cases} \tag{9}$$

then $\mathbb{P}(\lim_n \sum_{i=1}^n \phi_i g_x(x^*, \alpha_i) = +\infty) = 1$ *since* $\lim_n \frac{1}{n} \sum_{i=1}^n \alpha_i \ge \lim_n \frac{1}{n} \sum_{i=1}^n i = \lim_n \frac{n+1}{2} = +\infty$*. Similarly, we can have divergence to* $-\infty$*. However, for our purposes these situations are easily handled because we have* $\lim_n \mathbb{P}(S_n) = 0$ *irrespective of* $f_y(y^*)$*. Finally, we note that the existence of* $\lim_n \sum_{i=1}^n \phi_i g_x(x^*, \alpha_i)$ *is not assured (e.g. when the summation is oscillatory).*

Example 4 *Suppose* $g_x(x^*, \alpha_i)$ *are independent with mean* m_i *and variance* σ_i^2*. Consider a sequence of weights* ϕ_i *for which the assumption* $\sum_{i=1}^\infty \phi_i^2 = 0$ *does not hold: this rules out the case of equal weights* $\phi_i = 1/n$ *and can arise for example when a single agent has a non-vanishing weight (we deal with this case in Section 2.1). Assuming that* $m = \sum_{i=1}^\infty \phi_i m_i$ *and* $\sigma^2 = \sum_{i=1}^\infty \phi_i^2 \sigma_i^2$ *we have* $\mathbb{P}(C) = 1$ *due to Proposition 2 (i) and (ii). Also, as a consequence of the (Lyapunov) Central Limit Theorem, the almost sure limit for* $\sum_{i=1} \phi_i g_x(x^*, \alpha_i)$ *exists and is distributed as* $N(m, \sigma^2)$*. If* $f_y(y^*) > 0$ *we have*

$$\lim_n \mathbb{P}(S_n) = \int_{-1/f_y(y^*)}^{1/f_y(y^*)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-m)^2}{2\sigma^2}} dy \tag{10}$$

hence $\lim_n \mathbb{P}(S_n)$ *can take on any value between 0 and 1 and there is therefore no polarization in this case.*

2.1 Some generalizations and examples

We now extend the previous analysis to more general specifications of the problem.

[9] The convergence result in (8) still holds if the assumptions on the weights ϕ_i are weakened provided those on the random variables $\phi_i g_x(x^*, \alpha_i)$ are suitably strengthened.

[10] See [43], III.§6 for definition and characterizations.

1. To begin with, consider m-dimensional state variables in (6):

$$\begin{cases} \mathbf{x}_{t+1} = f(\mathbf{y}_{t+1}), \\ \mathbf{y}_{t+1} = \sum_{i=1}^{n} \phi_i g(\mathbf{x}_t, \alpha_i), \end{cases} \tag{11}$$

   where $f \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R}^m)$, $g \in \mathcal{C}^1(\mathbb{R}^{m+1}; \mathbb{R}^m)$. Assume $(\mathbf{x}^*, \mathbf{y}^*) \in \mathbb{R}^{2m}$ is the unique steady state of (11) irrespective of the α_i. Now, letting $\rho(\cdot)$ be the spectral radius of a square matrix, the stability of (11) depends on

$$\rho_n := \rho \left(J_f(\mathbf{y}^*) \sum_{i=1}^{n} \phi_i J_g(\mathbf{x}^*, \alpha_i) \right), \tag{12}$$

   where $J_f(\mathbf{y}^*)$ is the $m \times m$ Jacobian of f computed at $\mathbf{y}^*$ and $J_g(\mathbf{x}^*, \alpha_i)$ is an $m \times m$ random matrix whose entries are the partial derivatives of g computed at $(\mathbf{x}^*, \alpha_i)$. Note that ρ_n is a random variable and one could (with some effort) compute the probability $\mathbb{P}(\{\omega \in \Omega : \rho_n(\omega) < 1\})$ and analyze its limit.

2. Consider random vectors $\boldsymbol{\alpha}_i$, i.e. we are given a stochastic process where $\boldsymbol{\alpha}_i : \Omega \to A \subset \mathbb{R}^k$. The aggregated dynamical system appears the same as in (6). The stability sets S_n are as in (7) and the analysis goes along the same lines, bearing in mind that we have to use convergence results for random vectors. For example, Proposition 2, *(i)* and *(iii)*, still hold with straightforward changes.

3. Suppose the (scalar) state variable x evolves according to a law of motion which depends both on x and y. Specifically, system (6) becomes

$$\begin{cases} x_{t+1} = f(x_t, y_{t+1}) \\ y_{t+1} = \sum_{i=1}^{n} \phi_i g(x_t, \alpha_i) \end{cases} \tag{13}$$

   with the usual assumptions on the functions and the steady state. Aggregating, we study the one dimensional equation

$$x_{t+1} = f(x_t, \sum_{i=1}^{n} \phi_i g(x_t, \alpha_i)) \tag{14}$$

   whose stability reduces to

$$-1 < f_x(x^*, y^*) + f_y(x^*, y^*) \sum_{i=1}^{n} \phi_i g_x(x^*, \alpha_i) < 1 \tag{15}$$

   So we obtain the stability sets

$$S_n = \{\omega \in \Omega : f_y(x^*, y^*) \sum_{i=1}^{n} \phi_i g_x(x^*, \alpha_i(\omega)) \in (-1 - f_x(x^*, y^*), 1 - f_x(x^*, y^*))\} \tag{16}$$

   and we can perform the same asymptotic analysis we did in Section 2, simply adapting the endpoints of the convergence interval.

4. Consider the situation in which there are two distinct sets of state variables or, alternatively, distinct expectations of two pools of agents, described by the system

$$\begin{cases} x_{t+1} = f(x_t, y_{t+1}, z_{t+1}) \\ y_{t+1} = \sum_{i=1}^{n} \phi_i y_{t+1}^i \\ z_{t+1} = \sum_{j=1}^{m} \psi_j z_{t+1}^j \\ y_{t+1}^i = g(x_t, \alpha_i) \quad i = 1, \ldots, n \\ z_{t+1}^j = h(x_t, \beta_j) \quad j = 1, \ldots, m \end{cases} \tag{17}$$

   with scalar variables x, y^i, z^j and a unique steady state $(x^*, y^*, z^*) \in \mathbb{R}^3$. The stability analysis for (17) proceeds as above, aggregating heterogeneity through two sets of weights ϕ_i and ψ_j summing up to 1. Some details are provided in Example 5.

5. Finally, we focus on an interesting special case in which a particular agent has a significance superior to the others. Consider the problem

$$\begin{cases} x_{t+1} = f(y_{t+1}) \\ y_{t+1} = \phi_0 g(x_t, \alpha_0) + \sum_{i=1}^{n} \phi_i g(x_t, \alpha_i) \end{cases} \tag{18}$$

where the weights satisfy $0 \ll \phi_0 < 1$, $0 \leq \phi_i \leq 1 - \phi_0$, $\phi_0 + \sum_{i=1}^n \phi_i = 1$ for $n \geq 1$, $\phi_i = 0$ for $i > n$ and $\lim_{n\to+\infty} \phi_0 = \hat{\phi}_0 > 0$. This situation, whereby agent 0 has special influence in the economy, is in fact already encompassed in the analysis of Section 2. In addition, the presence of any finite number of influential agents can be easily accommodated. Assuming a unique steady state (x^*, y^*), we have

$$S_n = \{\omega \in \Omega : f_y(y^*)[\phi_0 g_x(x^*, \alpha_0(\omega)) + \sum_{i=1}^n \phi_i g_x(x^*, \alpha_i)(\omega)] \in (-1, 1)\}. \tag{19}$$

The asymptotic analysis for the probabilities of S_n readily follows. For instance, suppose $\alpha_0, \alpha_1, \alpha_2, \ldots$ are independent with α_i i.i.d. for $i \geq 1$ and $\phi_i = \frac{1}{n}(1 - \phi_0)$ for $1 \leq i \leq n$. Setting

$$\hat{\Omega} = \{\omega \in \Omega : \sum_{i=1}^n \phi_i g_x(x^*, \alpha_i)(\omega) \longrightarrow (1 - \hat{\phi}_0)\mathbb{E}[g_x(x^*, \alpha_1)]\}, \tag{20}$$

the SLLN implies $\mathbb{P}(\hat{\Omega}) = 1$. Now assume $f_y(y^*) > 0$ to fix ideas. Then

$$\lim_n \mathbb{P}(S_n \cap \hat{\Omega}) = \mathbb{P}\{\omega \in \hat{\Omega} : \alpha_0(\omega) \in g_x(x^*, \cdot)^{-1}(S)\} \tag{21}$$

where

$$S = \left(\tfrac{1}{\hat{\phi}_0}[-\tfrac{1}{f_y(y^*)} - (1 - \hat{\phi}_0)\mathbb{E}[g_x(x^*, \alpha_1)]]\ ,\ \tfrac{1}{\hat{\phi}_0}[\tfrac{1}{f_y(y^*)} - (1 - \hat{\phi}_0)\mathbb{E}[g_x(x^*, \alpha_1)]]\right) \tag{22}$$

Thus we can observe that polarization does not necessarily hold.

Example 5 *Inspired by the empirical literature devoted to estimating processes for the formation of households' stock market beliefs (see e.g. [47]) let us consider a financial market populated by two groups of traders, namely the* mean-extrapolators *and the* mean-reverters*. Let x_t be the current price of the traded asset and y^i_{t+1}, z^j_{t+1} its expected price at time $t+1$ (computed at time t) by the i-th and the j-th agent respectively. Suppose that the pricing rule as well as the expectation are (trivially) given by*

$$\begin{cases} x_{t+1} = a x_t + (1-a)\tau \sum_{i=1}^n \phi_i y^i_{t+1} + (1-a)(1-\tau)\sum_{j=1}^m \psi_j z^j_{t+1} \\ y^i_{t+1} = x^* + \alpha_i(x_t - x^*) \quad i = 1\ldots, n \\ z^j_{t+1} = x^* + \beta_j(x_t - x^*) \quad j = 1\ldots, m \end{cases} \tag{23}$$

where $a \in (0,1)$ is a constant market parameter, $\tau \in (0,1)$ measures the relative strength of the pools, x^* is the fundamental value of the asset, $\alpha_i > 1$ and $\beta_j < 0$ are the uncertain parameters which drive the expectations. With one trader per group, the probability of stability is $\mathbb{P}(S_1) \equiv \mathbb{P}(\{\omega \in \Omega : \tau\alpha_1(\omega) + (1-\tau)\beta_1(\omega) \in (-\frac{1+a}{1-a}, 1)\})$. Instead, assume an arbitrary number of investors and set $n = m$, $\phi_i = \psi_j = \frac{1}{n}$ to fix ideas. Moreover, we shall suppose α_i i.i.d. and β_j i.i.d., the two processes being independent each other. Now, when n tends to infinity we recover a polarization effect with almost sure stability if and only if $\tau\mathbb{E}[\alpha_1] + (1-\tau)\mathbb{E}[\beta_1] \in (-\frac{1+a}{1-a}, 1)$. For instance, fix $a = \frac{1}{4}$, $\tau = 0.8$, $\alpha_1 \sim U(1,2)$ and $\beta_1 \sim U(-1,0)$. Then, $\mathbb{P}(S_1) = \frac{3}{8}$, whereas in the limit we get instability since $\mathbb{E}[\alpha_1] = \frac{3}{2}$ and $\mathbb{E}[\beta_1] = -\frac{1}{2}$. Ceteris paribus, *setting $\tau = 0.6$ we have $\mathbb{P}(S_1) = \frac{11}{12}$ and stability in the limit.*[11] *More in general, $\tau = \frac{3}{4}$ implies a 50-50 probability of stability when there is one trader per group, and discriminates the asymptotic outcome when n becomes large. So, with this parameter specification, asymptotic convergence to the fundamental value requires a not too large weight of mean-extrapolators with respect to the mean-reverters.*

In the following example we revisit and expand the stylized financial market model originally introduced by [23] and make use of the results above.

Example 6 *Building on [23], active investors are segregated into two distinct pools, namely the α-investors (or* fundamentalists*) and the β-investors (or* sheep*). Fundamentalists tend to stabilize the market, whereas* sheep *amplify price oscillations. Each group is supposed to contribute to the price dynamics via an excess demand function, which in turn depends on several parameters. Once the aggregated excess demand Z_t is established, market makers clear the market and price adjusts in order to ensure equilibrium. The adjustment mechanism can be explicitly formalized as follows:*

$$x_{t+1} = x_t + \lambda Z_t(y_{t+1}, x_t),$$

[11] Observe that the same outcome continues to hold with more elaborate non-linear updating of the expectations, e.g. $y^i_{t+1} = x^* + \arctan(\beta_j(x_t - x^*))$ and/or $z^j_{t+1} = x^* + \beta_j(x_t - x^*) + \beta_j(x_t - x^*)^k$ for some $k > 1$.

where x is the price of the asset, y is the price expectation and $\lambda > 0$ is the adjustment sensitivity. For simplicity, we assume as in [19] that the excess demand is given by

$$Z_t(y_{t+1}, x_t) = s(y_{t+1} - x_t),$$

where $s > 0$ is a positive coefficient. We shall suppose that fundamentalists act as a whole. This is because α-investors drive the market thanks to their superior information, which we aggregate at this stage of the model. Otherwise stated, we assume the presence of a unique influential fundamentalist whose expectation is

$$y^0_{t+1} = x_t - \alpha_0 \left[(x_t - v) + (x_t - v)^3\right], \tag{24}$$

where v is an estimate of the investment value and $\alpha_0 > 0$ is a sensitivity parameter. For exhaustive explanations on (24)*, see [23, 19]. On the other hand, we keep separated n β-investors, whose expectations are*

$$y^i_{t+1} = x_t + \beta_i(x_t - u) \quad i = 1, \dots, n \tag{25}$$

where $\beta_i > 0$ are the so-called flocking coefficients and u is a perceived fundamental value. Before proceeding further, we assume $v = u$ as explained in [23] in order to consider only a full equilibrium. Next, we linearly aggregate expectations as usual, attaching weight ϕ_0 to the influential fundamentalist and weights ϕ_i to the sheep. *A weight simply represents the relative importance of the investor. Putting everything together, we get the equation*

$$x_{t+1} = x_t + \lambda s \left[(x_t - v)\sum_{i=1}^{n} \phi_i\beta_i - \alpha_0\phi_0(x_t - v) - \alpha_0\phi_0(x_t - v)^3\right], \tag{26}$$

The full equilibrium of (26) *corresponds to the steady state $x^* = v$. We distinguish the following interesting cases.*

Single sheep. *If $n = 1$ and α_0, β_1 are the stochastic parameters of our investors, we see that the probability of x^* being stable is $\mathbb{P}\left(\frac{1-\phi_0}{\phi_0}\beta_1 < \alpha_0 < \frac{2}{\lambda s\phi_0} + \frac{1-\phi_0}{\phi_0}\beta_1\right)$, where ϕ_0 is the constant weight associated to the fundamentalist. Such probability can be easily computed if the joint distribution of (α_0, β_1) is known. The analysis is consistent with [23], where the deterministic skeleton is thoroughly analyzed.*

Many sheep with same distribution. *Suppose β_i are i.i.d. with uniform weights $\phi_i = \frac{1}{n}(1 - \phi_0)$ for $i \leq n$. Hence, we can exploit the results in* (19)-(22) *and the resulting probability of stability is $P := \mathbb{P}\left(\frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1] < \alpha_0 < \frac{2}{\lambda s\phi_0} + \frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1]\right)$. For instance, if $\alpha_0 \sim U(0, \bar\alpha)$ we see that*

$$P = \begin{cases} 0 & \text{if} \quad \bar\alpha \leq \frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1] \\ 1 - \frac{1}{\bar\alpha}\frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1] & \text{if} \quad \bar\alpha \in \left(\frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1], \frac{2}{\lambda s\phi_0} + \frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1]\right) \\ \frac{1}{\bar\alpha}\frac{2}{\lambda s\phi_0} & \text{if} \quad \bar\alpha \geq \frac{2}{\lambda s\phi_0} + \frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1] \end{cases} \tag{27}$$

Note that P is increasing in $\bar\alpha$ (as long as $\bar\alpha \leq \frac{2}{\lambda s\phi_0} + \frac{1-\phi_0}{\phi_0}\mathbb{E}[\beta_1]$), since such parameter correlates with a higher impact of the fundamentalist on the market. P instead is decreasing in λ and s, both of which contribute to defining the responsiveness of price changes to excess demand.

3 Inertia in expectations

In the previous Section we have developed a model characterized by two features. From a technical point of view, the introduction of heterogeneity is absorbed within the dynamics of the system without affecting (increasing) its order. This in turn implies that polarization effects can be traced back to the action of a fictitious representative-agent in the spirit of [29]. In this Section we instead examine a family of models in which expectations may also depend on one's past values, i.e. exhibiting a component of inertia: this element is empirically relevant, as has been pointed out in the literature for example in [3, 6, 18, 33]. Incorporating such component however determines the dependence of the order of the dynamical system - and therefore the complexity of its analysis - on the number of agents involved in the economic activity.

We can represent such a scenario in the baseline model of a single-agent economy through the discrete-time dynamical system

$$\begin{cases} x_{t+1} = f(y_{t+1}) \\ y_{t+1} = g(x_t, y_t, \alpha_1) \end{cases} \tag{28}$$

with the usual assumptions that f and g are C^1, and $\alpha_1 : \Omega \to A \subset \mathbb{R}$ is a scalar random variable. Assume $(x^*, y^*) \in \mathbb{R}^2$ is the unique steady state of (28), i.e., $x^* = f(y^*)$ and $y^* = g(x^*, y^*, \alpha)$ for all $\alpha \in A$. To analyze the stability of the equilibrium, we consider the characteristic polynomial of the random 2×2 matrix

$$J_1 = \begin{bmatrix} f_y(y^*)g_x(x^*, y^*, \alpha_1) & f_y(y^*)g_y(x^*, y^*, \alpha_1) \\ g_x(x^*, y^*, \alpha_1) & g_y(x^*, y^*, \alpha_1) \end{bmatrix}. \tag{29}$$

We need to compute the probability that the spectral radius of J_1 lies in $(0, 1)$. Given the first row is a multiple of the second, we see that an eigenvalue of J_1 is $\lambda_1 = 0$, therefore we can compute the second eigenvalue as $\lambda_2 = f_y(y^*)g_x(x^*, y^*, \alpha_1) + g_y(x^*, y^*, \alpha_1)$. Hence, stability requires $-1 < \lambda_2 < 1$.

The model extends to the n agents case very much as above. The i-th agent is assumed to update her expectation about the state variable via the mapping

$$y^i_{t+1} = g(x_t, y^i_t, \alpha_i). \tag{30}$$

In other words, the i-th agent disregards (or is unaware of) the previous predictions of other agents when updating her own. Aggregating through the usual linear combination with weights ϕ_i as above, results in the $(n + 1)$-dimensional dynamical system:

$$\begin{cases} x_{t+1} = f\left(\sum_{i=1}^n \phi_i g(x_t, y^i_t, \alpha_i)\right) \\ y^i_{t+1} = g(x_t, y^i_t, \alpha_i) \quad i = 1, \ldots, n. \end{cases} \tag{31}$$

We thus see that the unique steady state of (31) is $(x^*, y^*, \ldots, y^*) \in \mathbb{R}^{n+1}$ (with an abuse of notation, we will sometimes write (x^*, y^*) instead).

Stability of system (31) depends on the spectral radius of the $(n + 1) \times (n + 1)$ random matrix

$$J_n = \begin{bmatrix} f_y(y^*)\sum_{i=1}^n \phi_i g_x(x^*, y^*, \alpha_i) & f_y(y^*)\phi_1 g_y(x^*, y^*, \alpha_1) & \cdots & f_y(y^*)\phi_n g_y(x^*, y^*, \alpha_n) \\ g_x(x^*, y^*, \alpha_1) & g_y(x^*, y^*, \alpha_1) & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ g_x(x^*, y^*, \alpha_n) & 0 & \cdots & g_y(x^*, y^*, \alpha_n) \end{bmatrix} \tag{32}$$

Otherwise stated, we have

$$S_n = \{\omega \in \Omega : \rho(J_n(\omega)) \equiv \max_{1 \le i \le n+1} |\lambda_i(\omega)| < 1\}, \tag{33}$$

where $\lambda_1, \ldots, \lambda_{n+1}$ are the eigenvalues of J_n. Now, let $p_n(\lambda)$ be the characteristic polynomial of J_n. Observe that p_n is a random polynomial of degree $(n + 1)$, meaning that the coefficients of p_n are random variables. However, for any fixed $\omega \in \Omega$, p_n is not random and its properties are as follows.

Lemma 7 *Assume the terms $g_x(x^*, y^*, \alpha_i)$ all have the same sign. Then:*

(i) $p_n(\lambda) = -\lambda \left[f_y(y^*) \sum_{i=1}^n \phi_i g_x(x^*, y^*, \alpha_i) \prod_{j \ne i}(g_y(x^*, y^*, \alpha_j) - \lambda) + \prod_{j=1}^n (g_y(x^*, y^*, \alpha_j) - \lambda) \right]$

(ii) p_n has $(n + 1)$ real roots

(iii) At least $n - 1$ roots of p_n belong to the interval $[\min_i\{g_y(x^*, y^*, \alpha_i)\}, \max_i\{g_y(x^*, y^*, \alpha_i)\}]$

Lemma 7 (iii) implies that if the realization of the $g_y(x^*, y^*, \alpha_i)$ terms all lie in the $(-1, 1)$ interval then p_n will have (at least) $n - 1$ stable roots. It turns out that the condition $\mathbb{P}(\{|g_y(x^*, y^*, \alpha_i)| > 1\}) = 0$ is also necessary to have asymptotic stability for large n, as we show in the following Proposition.

Proposition 8 *Let $A_i = \{\omega \in \Omega : |g_y(x^*, y^*, \alpha_i(\omega))| > 1\}$ and assume α_i are independent random variables. If $\sum_{i=1}^\infty \mathbb{P}(A_i) = \infty$, then $\lim_n \mathbb{P}(S_n) = 0$. In particular, if α_i are identically distributed and $\mathbb{P}(A_1) > 0$, then $\lim_n \mathbb{P}(S_n) = 0$.*

We shall henceforth assume α_i i.i.d. and such that $\mathbb{P}(\{|g_y(x^*, y^*, \alpha_i)| > 1\}) = 0$, i.e. the support of $g_y(x^*, y^*, \alpha_i)$ is included in the interval $[-1, 1]$. The question of stability remains open however, since it is still necessary to study the magnitude of the (at most two) remaining eigenvalue(s). Generally speaking, it is difficult to draw precise conclusions about them. However, there are some tractable special cases which are interesting as they include or are quite close to economic models such as production cobweb models with adaptive agents (see for example [34, 39, 20]) with applications to farming production cycles.

Proposition 9 *Assume α_i are i.i.d., $\lim_n \sum_{i=1}^n \phi_i^2 = 0$, $g_x(x^*, y^*, \alpha_i)$ have constant sign for all i and $g_y(x^*, y^*, \alpha_i) \in (-1, 1)$ similarly have constant sign. Then*

(i)

$$f_y(y^*)g_x(x^*, y^*, \alpha_i) > 0 \Rightarrow \lim_n \mathbb{P}(S_n) = \begin{cases} 1 & \text{if } \mu f_y(y^*) + 1 > 0 \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

where $\mu = \mathbb{E}\left[\frac{g_x(x^*, y^*, \alpha_1)}{g_y(x^*, y^*, \alpha_i) - 1}\right]$

(ii)

$$f_y(y^*)g_x(x^*, y^*, \alpha_i) < 0 \Rightarrow \lim_n \mathbb{P}(S_n) = \begin{cases} 1 & \text{if } \nu f_y(y^*) + 1 > 0 \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

where $\nu = \mathbb{E}\left[\frac{g_x(x^*, y^*, \alpha_i)}{g_y(x^*, y^*, \alpha_i) + 1}\right]$.

The assumption *(i)* of Proposition 9 implies that f drives y to the steady state y^* monotonically. On the contrary, in case *(ii)* the law of motion f follows a cobweb-like dynamic.

Remark 10 *Observe that there is a similarity between the conditions identifying the thresholds for stability in Proposition 9 and that for the single-agent case concerning the matrix in* (29). *However, given the nonlinearity of the expectation operator and the functional shape of μ and ν in Proposition 9, in general the average representative agent model requires conditions that differ from the n-agents scenario: so this makes a case for the concept that heterogeneous agents matter. The following Example illustrates this point.*

Example 11 *Our setting is well suited to represent the cobweb production model with heterogeneous producers studied, e.g., in [34, 20]. Indeed, consider the system*

$$\begin{cases} x_{t+1} = D^{-1}(\sum_{i=1}^n \phi_i S(y^i_{t+1})) \\ y^i_{t+1} = y^i_t + \alpha_i(x_t - y^i_t), \quad i = 1, \dots, n \end{cases} \tag{36}$$

where x is the current price of the commodity, y^i is the expected price for the i-th agent, D and S are the demand and supply function respectively, ϕ_i reflects the market share of the i-th agent and α_i are i.i.d. random variables with support on the unit interval. Note that every producer has adaptive expectations over the price. Assuming a strictly monotone decreasing demand and a strictly monotone increasing supply, we deduce the existence of a unique steady state $(x^*, y^*, \dots, y^*) \in \mathbb{R}^{n+1}$, where $x^* = y^*$ is the price that induces zero excess demand. Observe that, in terms of our previous notation, $f_y(y^*) = \frac{S'(x^*)}{D'(x^*)} < 0$, $g_x(x^*, y^*, \alpha_i) = \alpha_i > 0$ and $g_y(x^*, y^*, \alpha_1) = 1 - \alpha_i$. We can therefore apply Proposition 9, (ii), *letting $\nu = \mathbb{E}\left[\frac{\alpha_i}{2-\alpha_i}\right]$: asymptotic stability will emerge (for large n) if*

$$\mathbb{E}\left[\frac{\alpha_i}{2 - \alpha_i}\right]\frac{S'(x^*)}{D'(x^*)} + 1 > 0 \tag{37}$$

Observe that, by virtue of Jensens' Inequality, $\mathbb{E}\left[\frac{\alpha_i}{2-\alpha_i}\right] > \frac{\mathbb{E}(\alpha_i)}{2-\mathbb{E}(\alpha_i)}$ which entails that in this case a representative agent having parameter α equal to the population mean would imply weaker conditions for stability than those required by (37) *for the fully heterogeneous case. To inspect further the condition posed by* (37) *we can rewrite it in terms of elasticities of demand and supply at the equilibrium*

$$\frac{\epsilon_D}{\epsilon_S} < \mathbb{E}\left[\frac{\alpha_i}{2 - \alpha_i}\right]$$

Therefore we see that, given expectations (the exact probability distribution of α_i) and technology as captured by the elasticity of supply, demand must be inelastic enough in order for stability to arise in the presence of a large number of producers with heterogeneous adaptive expectations.

3.1 Some generalizations and examples

The first extension to (31) concerns a more general form for the law of motion f. We consider the system

$$\begin{cases} x_{t+1} = f(x_t, \sum_{i=1}^n \phi_i g(x_t, y^i_t, \alpha_i)) \\ y^i_{t+1} = g(x_t, y^i_t, \alpha_i), \quad i = 1, \dots, n \end{cases} \tag{38}$$

where f is allowed to depend directly on the current state x (in addition to its indirect influence through expectations). Adapting the arguments of the baseline model with inertia, the following result shows that the direct dependence on the state as measured by the partial derivative f_x at the steady state is key to determine the scope for stability with large n.

Proposition 12 *Under the assumptions of Proposition 9 and that $f_y(x^*, y^*)g_x(x^*, y^*, \alpha_i)g_y(x^*, y^*, \alpha_i) > 0$*

$$\lim_n \mathbb{P}(S_n) = \begin{cases} 1 & \text{if} \quad -1 - \nu f_y(x^*, y^*) < f_x(x^*, y^*) < 1 + \mu f_y(x^*, y^*) \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

where μ and ν are defined in Proposition 9 and S_n are the stability sets of (38).

Remark that Proposition 12, differs from Proposition 9 in that (39) involves μ and ν simultaneously.for

Example 13 *Consider an asset pricing model characterized by a type of agents who form their own expectation y^i about the future price combining an adaptive rule and a fundamental analysis over the traded asset (see [19, 23] for similar settings). Specifically, this financial market is described by the system*

$$\begin{cases} x_{t+1} = x_t + \delta(y_{t+1} - x_t) \\ y^i_{t+1} = \alpha_i y^i_t + (1 - \alpha_i)x_t - \beta_i[(x_t - x^*) + (x_t - x^*)^3], \quad i = 1, \dots, n \end{cases} \tag{40}$$

with the usual linear aggregation of expectations $y_t = \sum_i^n \phi_i y^i_t$. Here x^* is an estimate of the fundamental value of the asset and $\delta > 0$ is a sensitivity parameter which depends on the tatônnement process underlying the pricing rule (see [19, 23] for more details). We shall assume independently distributed $\alpha_i \in (0, \bar{\alpha})$ and $\beta_i \in (0, 1 - \bar{\alpha})$, with $\bar{\alpha} \in (0, 1)$. Observe that the stochastic parameter is bi-dimensional in this case but this does not compromise our results. The unique steady state of (40) *is $(x^*, x^*, \dots, x^*) \in \mathbb{R}^{n+1}$, $f_x(x^*, y^*) = 1 - \delta$, and $f_y(x^*, y^*) = \delta$.*

If we consider a single-agent model with given parameters (α, β), then we see that stability holds for $\delta < \frac{2+2\alpha}{2\alpha+\beta}$ and period-doubling bifurcations can occur, whereas when $\delta < 2$ stability is always ensured. In the general case, assuming (α_i, β_i) i.i.d., we can apply Proposition 12. For example, if $\alpha_i \sim U(0, \bar{\alpha})$ and $\beta_i \sim U(0, 1 - \bar{\alpha})$, then we find

$$\mu = \frac{(\bar{\alpha} - 1)\ln(1 - \bar{\alpha})}{2\bar{\alpha}} - 1, \qquad \nu = \frac{(3 + \bar{\alpha})\ln(1 + \bar{\alpha})}{2\bar{\alpha}} - 1.$$

As a consequence, when $\delta \le 1$ we always have stability. On the contrary, if $\delta \ge 4$ then stability can never arise. Finally, when $1 < \delta < 4$ we see that

$$\lim_n \mathbb{P}(S_n) = 1 \iff \delta < \frac{4\bar{\alpha}}{4\bar{\alpha} - (3 + \bar{\alpha})\ln(1 + \bar{\alpha})}, \tag{41}$$

where S_n are the stability sets of (40)*. Observe that the stability threshold appearing in the right-hand side of* (41) *is monotone decreasing in $\bar{\alpha}$: the larger the support for the adaptive parameter - that is strengthening the weight of the adaptive component with respect to the fundamental component of expectations, the condition for stability becomes more restrictive. In other words, the more agents there are with relevant adaptive components, the less likely it is that with n large the system will converge to steady state.*

A second extension considers the case when the law of motion f depends on the current value of y rather than on its value at $t + 1$. Such setup is quite natural in many applications.[12] Hence, we are led to study the system

$$\begin{cases} x_{t+1} = f(x_t, \sum_{i=1}^n \phi_i y^i_t) \\ y^i_{t+1} = g(x_t, y^i_t, \alpha_i), \quad i = 1, \dots, n \end{cases} \tag{42}$$

The arguments in the base case can be tweaked in a straightforward way to tackle this case and give the following result.

Corollary 14 *Under the assumptions of Proposition 9 and that $f_y(x^*, y^*)g_x(x^*, y^*, \alpha_i) > 0$*

$$\lim_n \mathbb{P}(S_n) = \begin{cases} 1 & \text{if} \quad -1 + \nu f_y(x^*, y^*) < f_x(x^*, y^*) < 1 + \mu f_y(x^*, y^*) \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

where μ and ν are defined in Proposition 9 and S_n are the stability sets of (42).

[12] System (42) indeed can be easily cast into a simple two-dimensional Vector AutoRegressive model of order 1, which is widely used in empirical and financial macroeconomics (see [44]). Also, it fits for instance a nonlinear discrete-time Cournot duopoly game in which sellers have heterogeneous expectations, as in [5] and many other papers.

Notice how (43) differs from (39): this stems from the underlying Jacobian not being singular any more and its characteristic polynomial differing accordingly. Specifically, (43) gives a stronger condition on $f_x(x^*, y^*)$ in order to have asymptotic stability, since $\nu f_y(x^*, y^*) > 0$. So in general we can expect tighter conditions to be required to secure stability for large n.

Example 15 *An R&D consortium launches a new high-tech product on the market, which requires heavy investment by manufacturing companies, with the support of the Government. The initial investment guarantees at least an aggregate production of $\bar{x}$ units, with a minimal rate of use of the plants and the other factors by each firm. The producers cannot engage in a Bertrand competition since the consortium rules bind the companies to a given selling price depending on aggregate production, which in turn generates profit margins for the firm. For this reason firms focus on the aggregate production x, their share of profits thereof, $\delta^i x$, and the individual extra-production with respect to $\bar{x}$: to this end firms can push the use of their plants and other inputs beyond the minimum capacity, as measured by y_i. The following system of equations depicts the above scenario:*

$$\begin{cases} x_{t+1} = \bar{x} + \rho \sum_{i=1}^{n} \phi_i y_t^i \\ y_{t+1}^i = \alpha_i y_t^i + \delta_t^i x_t, \quad i = 1, \dots, n \end{cases} \tag{44}$$

were we assume $\rho > 0, \alpha_i \in (0, 1)$. Thus firms contribute to producing above the minimum threshold $\bar{x}$ in a way that depends on previous production: if the profit margins were zero ($\delta_t^i = 0$) they would gradually reduce (at a rate $0 < \alpha_i < 1$) their extra production, reflecting constraints due to existing contracts by which they could not immediately zero out y^i. Notice that firms capacity utilization rate above the minimum translates into quantities produced in a way that depends on the factor $\rho\phi_i$, which relates to the firm's individual efficiency. For simplicity we also assume that there is no actual idiosyncratic profit component, so $\delta_t^i = \delta > 0$.
With a single producer, there is a meaningful (in the sense of securing the non-negativity of the variables) unique steady state of (44) provided $\rho\delta < 1 - \alpha$, and it is as follows

$$\left(\frac{\bar{x}(1-\alpha)}{1-\alpha-\rho\delta}, \frac{\bar{x}\delta}{1-\alpha-\rho\delta} \right)$$

Notice that the condition $\rho\delta < 1 - \alpha$ ensures that the steady state level of production is larger than the minimal threshold, and at the same time it is asymptotically stable. If for example $\alpha \sim U(\alpha_l, \alpha_h)$, then we find

$$\mathbb{P}(\{\omega \in \Omega : \rho\delta < 1 - \alpha(\omega)\}) = \begin{cases} 0 & \text{if } 1 - \rho\delta \le \alpha_l \\ \frac{1-\rho\delta-\alpha_l}{\alpha_h - \alpha_l} & \text{if } \alpha_l < 1 - \rho\delta < \alpha_h \\ 1 & \text{if } 1 - \rho\delta \ge \alpha_h \end{cases} \tag{45}$$

On the other hand, with n producers there is a unique steady state which depends on the realization of the α_i as follows:

$$x^* = \frac{\bar{x}}{1 - \rho\delta \sum_{i=1}^{n} \frac{\phi_i}{1-\alpha_i}}, \quad y^{i*} = \frac{\delta x^*}{1-\alpha_i}, \quad i = 1, \dots, n.$$

Assuming α_i i.i.d. and for example equal weights $\phi_i = \frac{1}{n}$ allow us to resort to Corollary 14. Observing that $f_x(x^*, y^*) = 0$, $f_y(x^*, y^*) = \rho$, we find asymptotic stability if and only if

$$\rho\delta < \left(\mathbb{E}\left[\frac{1}{1-\alpha_1} \right] \right)^{-1}$$

For instance, if $\alpha_1 \sim U(\alpha_l, \alpha_h)$, then asymptotic stability arises under the condition

$$\rho\delta < \frac{1-\alpha_l}{\ln(1-\alpha_l) - \ln(1-\alpha_h)}$$

Let us compare this result with condition (45) arising in the single-firm case. One obvious point is that, given there is polarization, stability with a large number of firms ceases to be stochastic and either occurs or does not. This is to be contrasted with the single-firm situation (45) which involves an intermediate regime of probability strictly between 0 and 1: however if we focus on the condition that entails stability with certainty, the constraint is weaker with large n: a government-supported R&D consortium therefore would probably weigh this along with other aspects, such as optimal partner size in order to determine the ideal composition and membership conditions of the consortium.

4 Discussion and possible extensions

In the previous sections, we sought to understand the extent to which the study of the local dynamic properties of a model with heterogeneous agents can be traced back to that of the corresponding model with an average representative agent. This was done by envisioning that the heterogeneous economy is populated by a very large number of types drawn from a reference population whose probability distribution is known. On the one hand, we have shown that an asymptotic result relating the stability properties of the heterogeneous model to certain characteristics of the population distribution is valid under fairly general assumptions. In spirit, this result is close to the one obtained in [13], with the drawback of being related only to the local dynamical properties of the model, but with the benefit of being applicable to a larger set of cases. When heterogeneity increases, we find that the local stability of the system is assured (in probability) if and only if the probability distribution satisfies conditions that depend on the structure of the model under study. On the other hand, we observe that such conditions cannot always be trivially connected to some average measure of the distribution of types, particularly when agents heterogeneity incorporates some kind of adaptation with inertia, as seen in Section 3.

While we have tried to cover a sufficiently wide set of different basic models in order to make a convincing point about the general validity of our two main sets of results, many extensions are possible which would contribute to widely expand the scope of this work. We will now discuss some of them, trying to highlight the role played by the assumptions on which our results are based.

One direction in which the assumptions can be relaxed concerns the probabilistic structure on which the model rests: for instance we might relinquish the fact that α should have first and second moments. An example is the following, which considers a possible twist of Example 6.

Example 16 *We investigate the case of many sheep with different distributions in Example 6. Assume we are given (possibly distinct) weights ϕ_i summing up to $1-\phi_0$ and independent random variables β_i with Lévy distribution, which has support $[0,+\infty)$ and is described by a single parameter $b_i > 0$; we shall write $\beta_i \sim L(b_i)$ and its density function is unimodal, with mode at $x = \frac{b_i}{3}$ and $\mathbb{E}[\beta] = +\infty$. Also Lévy distributed random variables are infinitely divisible: indeed*

$$\sum_{i=1}^{n} \phi_i \beta_i \sim L\left(\left[\sum_{i=1}^{n} \sqrt{\frac{b_i}{\phi_i}}\right]^2\right).$$

It follows that if $\lim_n \sum_{i=1}^{n} \sqrt{\frac{b_i}{\phi_i}} =: b < +\infty$, then we can perform our stability analysis in the limit for n growing to infinity. Note that the assumption $b < +\infty$ gives a quite strict condition on the parameters b_i and the weights ϕ_i. For example, if $\phi_i = 1/n$, then the b_i must be $o(n^{-3})$. Now, if α_0 is known we have

$$\lim_n \mathbb{P}(S_n) = \begin{cases} 2\left[N\left(\sqrt{\frac{b}{-2/\lambda s+\alpha_0\phi_0}}\right) - N\left(\sqrt{\frac{b}{\alpha_0\phi_0}}\right)\right] & \text{if } -\frac{2}{\lambda s} + \alpha_0\phi_0 > 0 \\ 2\left[1 - N\left(\sqrt{\frac{b}{\alpha_0\phi_0}}\right)\right] & \text{otherwise} \end{cases} \tag{46}$$

where N is the cumulative density of a standard Gaussian random variable. Thus polarization does not arise. Furthermore, as long as $-\frac{2}{\lambda s} + \alpha_0\phi_0 \le 0$, the asymptotic probability in (46) is monotone increasing in α_0 and ϕ_0, whereas it is decreasing in b and λs as expected. Surprisingly, when $-\frac{2}{\lambda s} + \alpha_0\phi_0 > 0$ the probability is monotone decreasing in λs but it is not monotone with respect to any of the other parameters. Before concluding this example, we remark that Proposition (2), (ii), just gives a sufficient condition in order to have $\mathbb{P}(C) = 1$. As a matter of fact, we show in the Appendix that if $b < +\infty$, then $\mathbb{P}(\omega \in \Omega : \sum_{i=1}^{n} \phi_i\beta_i$ converges$) = 1$ even if $\mathbb{E}[\beta_i] = +\infty$ for all i. Clearly, the almost sure limit of the summation must be a Lévy distributed random variable with parameter b.

In addition to various technical aspects, possible extensions can be organised in distinct substantive areas as we examine below.

4.1 Coexistence of positive and negative feedback

While the main result of Proposition 2 in Section 2 holds under fairly weak assumptions, its extension to the more general framework of Section 3 hinge upon the crucial sign assumptions of Proposition 9. These technical assumptions rule out the possibility to apply those results to interesting settings some of which are discussed below. The assumption about g_x excludes models in which two different groups of agents form expectations at a given time using previous expectations (inertia) and respond to changes in the state variable in opposite directions.

One such situation is studied in [19] who suppose that agents are a heterogeneous mixture of sophisticated and unsophisticated behavior. They assume expectations are generated by way of the law

$$p_{t+1}^{e}=\alpha p_{t}^{e}+(1-\alpha)\,p_{t}-\beta\left(\left(p_{t}-v\right)+\left(p_{t}-v\right)^{3}\right)+\gamma\left(p_{t}-p_{t-1}\right) \tag{47}$$

which, for extreme values of the parameters α, β and γ reduces to a version of the Day and Huang model.[13] In this case inertia may go along with reactions to price movement of both signs, hence revealing an example which is not covered in this paper.

A second group of models left out from our analysis are those described by

$$p_{t+1}^{e}=\alpha+\sum_{i}^{L}\beta_{i}p_{t-i}+\sum_{i}^{L}\gamma_{i}p_{t-i}^{e} \tag{48}$$

when there are no constraints on the parameters β_i, γ_i. Versions of such prediction rules have been used (and estimated) in [36, 33, 6, 3] in the context of lab experiments. It is however interesting to remark that, in [33], where estimations are carried out based on experimental data, our sign conditions (either β_i or γ_i having different sign across the estimated sample) were not violated.

Further examples include [21] where an inflation targeting monetary policy model generates expectations-driven inflation dynamics with both the Central Bank and the private sector playing a role thereof. This case is somewhat trickier because changes in expectations from the Central Bank and the private sector have opposite effects on the state variable, whilst both expectations mechanisms feature positive inertia. The structure of the model is:

$$\begin{cases} x_{t+1}=f(x_t, y_t, \sum_{i=1}^{n}\phi_i z_t^i) \\ y_{t+1}=g(x_t,y_t) \\ z_{t+1}^{i}=h(x_t,z_t^i,\alpha_i), \quad i=1,\ldots,n \end{cases} \tag{49}$$

with $g_x, g_y, h_x, h_z > 0$ as required, but $f_y f_z < 0$ hence implying that the results in this paper do not shed light on such model when heterogeneity is introduced.[14]
However, in [19, 21] the authors numerically show that the heterogeneous models of equations (47) and (49) cannot be reduced to the corresponding average representative-agent case, and that a multiplicity of polarization outcomes emerge when the number of agents increases, which is suggestive that the results presented here may carry through to such models.

4.2 Aggregation

Across the paper we have so far assumed that heterogeneity is aggregated through a weighted average of individual choices within the law of motion of the state variable of interest. While this is natural in many contexts, there are cases where this assumption poses excessively demanding constraints. An example is given by the following model of a market with one dominant firm and a competitive fringe (see e.g. [8, 9]).

Example 17 *Let's consider the market for a good with a linear demand function $p = a - bq$, and a supply side of the market populated by one big, rational, profit maximizing firm, and a fringe of small, naive, price taking potential entrants. A public authority monitors price dynamics and implements policies to push prices towards a target p^*. Small firms have a total cost function $CT_i = \frac{(q^i)^2}{2\gamma_i} + F_i$ and once entered the market they each produce a quantity q^i such that price and marginal cost are equal, $\frac{q^i}{\gamma_i} = p \Rightarrow q^i = \gamma_i p$. At each point in time t firms draw their production decisions for the following period given their price expectations, which are as follows:*

$$p_{t+1}^{i}=p^{*}+\alpha_{i}\left(p_{t}-p^{*}\right)$$

If there are n small firms in the market, then the total quantity they produce at time $t+1$ is

$$q_{t+1}^{S}=\sum_{i=1}^{n}q^{i}=\sum_{i=1}^{n}\gamma_{i}p_{t+1}^{i}=\sum_{i=1}^{n}\gamma_{i}\left(p^{*}+\alpha_{i}\left(p_{t}-p^{*}\right)\right)=p^{*}\sum_{i=1}^{n}\gamma_{i}\left(1-\alpha_{i}\right)+p_{t}\sum_{i=1}^{n}\gamma_{i}\alpha_{i}$$

[13]Observe that standard asset pricing models with fundamentalists and chartists do not belong to this category, and are covered by the discussion in Section 2, because expectations - despite assuming agents who react in opposite way when prices depart from the equilibrium value - do not embody inertia. The classical paper by Day and Huang [23] provides a trickier example. They consider an asset pricing model where the market is populated by two types of agents, *sophisticated* (i.e. fundamentalists) and *unsophisticated* who forecast future prices as a mixture of adaptive and trend chasing expectations. However such version of the model is encompassed in Section 3 because the inertia in the model is related to agents having all the same kind of reaction to price movement thus satisfying the assumption of g_x all having the same sign.

[14]Instead, the result in Proposition 9 could be easily adjusted to encompass the case $f_y f_z > 0$.

The big firm anticipates small firms behavior and acts as a monopolist on the residual demand. Given its total cost function $CT = cq^B + F$ profit maximization implies

$$a - 2bq^B - bq^S - c = 0 \Rightarrow q^B = \frac{a-c}{2b} - \frac{q^S}{2}$$

and the dynamics of the system is described by the following set of equations

$$\begin{cases} p_{t+1} = a - b\left(q^B_{t+1} + q^S_{t+1}\right) \\ q^S_{t+1} = \sum_{i=1}^n q^i_{t+1} \\ q^B_{t+1} = \frac{a-c}{2b} - \frac{q^S_{t+1}}{2} \\ q^i_{t+1} = \gamma_i p^i_{t+1} \qquad i = 1 \dots n \\ p^i_{t+1} = p^* + \alpha_i \left(p_t - p^*\right) \qquad i = 1 \dots n \end{cases} \Rightarrow p_{t+1} = \frac{a+c}{2} - \frac{b}{2}\sum_{i=1}^n \gamma_i \left(p^* + \alpha_i \left(p_t - p^*\right)\right) \tag{50}$$

Thus, in this example it is natural to aggregate heterogeneity through a simple sum and not through a weighted average. This fact is the source of several dissimilarities when compared to what we have seen so far, the most important of which is the fact that the steady state depends on the magnitude and quality of heterogeneity. Nonetheless, with some caution, the equilibrium analysis can be carried out as usual as we will see now.

Example 17 (cont'd) *The system described by equation* (50) *has the steady state $\hat{p}$:*

$$\hat{p} = \frac{a + c - bp^* \sum_{i=1}^n \gamma_i \left(1 - \alpha_i\right)}{2 + b\sum_{i=1}^n \gamma_i \alpha_i} \tag{51}$$

which depends on several parameters and, in particular, on the size n and types γ_i, α_i of the competitive fringe. In general, the announced price p^* will not be consistent with the steady state, so the public authority will need to consider the effect of the chosen target on the value of the equilibrium that will be actually achieved as well as on its stability. If it aims at having a steady state of the system coherent with the target itself, given the structure of market demand, and given the number n of small firms, the public authority should announce the following target price

$$p^* = \frac{a+c}{2+b\Gamma}$$

where $\Gamma = \sum_{i=1}^n \gamma_i$. In this situation, assuming that in equilibrium the total quantity produced by small firms does not exhaust demand and that profits are non-negative for all firms in the market, the stability condition $-1 < \left.\frac{dp_{t+1}}{dp_t}\right|_{p_t = p^*} < 1$ requires that

$$\sum_{i=1}^n \frac{\gamma_i}{\Gamma}\alpha_i < \frac{2}{b\Gamma}$$

so stability is granted if the weighted average of the behavioral parameters α_i with weights equal to $\frac{\gamma_i}{\Gamma}$, representing a measure of the fraction of small-firms total quantity produced by firm i, is less than $\frac{2}{b\Gamma}$.
Any different target p^* would imply the steady state described by equation (51) *which is decreasing in p^*. The intuition is that if the target is high, small firms will adapt their expectations to it and hence increase production, hence leading to a price decline. Anyway, a public authority willing to avoid any credibility issue which might stem from the inconsistency between announced and realized price should announce a target price of $p^* = \frac{a+c}{2+b\Gamma}$ and, possibly, try to influence it by favouring or thwarting the entrance (or permanence) of small firms in the market - hence manipulating the parameter Γ. Remark that the larger the number of small firms, the greater Γ, and the smaller p^*. So, in this regard, the public authority should favour the entrance of the largest possible number of small firms. When the number of small firms becomes larger and larger, the stability condition reduces to*

$$E\left[\alpha_i\right] < \frac{2}{b\hat{\Gamma}} \qquad \text{with} \quad \hat{\Gamma} = \lim_{n\to+\infty}\Gamma$$

following from the Kolmogorov's strong law of large numbers. As a consequence, the increase in the number of small firms in the market pushes down the price and, at the same time, makes it harder for the stability conditions to be satisfied, generating a dilemma for the public authority.

4.3 Endogenous weights and learning

One assumptions we have maintained is that the weight of each agent (or firm), ϕ_i remains constant in time. Relaxing such assumption within the above framework, could shed some light on situations where weights may become endogenous and possibly reflect some underlying process of learning.

An example can be obtained observing that system (50) from Example 17 can be recast as

$$\begin{cases} p_t = a - b\left(q_t^B + q_t^S\right) = a - b\left(\phi_t^0 q_t + \sum_{i=1}^n \phi_t^i q_t\right) \\ q_t = \frac{a-c}{2b} + \frac{\sum_{i=1}^n \gamma_i p_t^i}{2} \\ p_{t+1}^i = p^* + \alpha_i\left(p_t - p^*\right) \\ \phi_t^i = \frac{\gamma_i p_t^i}{q_t} \\ \phi_t^0 = 1 - \sum_{i=1}^n \phi_t^i \end{cases}$$

Written in this way the weights ϕ^i sum up to one but are time-varying. Models in which weights change in time may be complex and, except specific cases such as the above, are outside the scope of this work: examples include, within the framework of Example 17, the possibility of having entry and exits of fringe firms based on expected profits and losses.

Further examples worth studying would have weights that reflect the relative performance of given predictors, in the spirit of the *adaptive rational equilibrium dynamics* (A.R.E.D., [11] is the classical reference) and more in general related to the literature about endogenous switching between predictors. The basic model introduced by Brock and Hommes in [11] has the form

$$x_{t+1} = f\left(x_t, \sum_{i=1}^{2} \phi_t^i y_t^i\right)$$

in which y_t^1, y_t^2 are two different expectation functions and the weights are

$$\phi_t^i = \frac{e^{-\beta h_i(x_t, y_t^i)}}{e^{-\beta h_1(x_t, y_t^1)} + e^{-\beta h_2(x_t, y_t^2)}}$$

where the functions h_i represent a measure of fitness of predictors y^i and β is the intensity of choice determining the rate of adjustment towards the best predictor.

Extending these models to the case of a large number of different heterogeneous agents leads to a more complex situation. The *large type limit* model introduced in [13] considers the case of a generic market model with heterogeneous sellers and buyers. Selling and buying decisions are adjusted in time according to an ex-post evaluation of the performance of a given set of alternative functions (each characterized by the realization of a random variable). In its simplest version, the model takes the reduced form

$$F_{I,J}(x_{t+1}, \mathbf{x}_t, \alpha_i, \beta_j) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \phi_{i,j}(\mathbf{x}_t, \alpha_i, \beta_j) G(x_{t+1}, \mathbf{x}_t, \alpha_i, \beta_j)$$

where $\mathbf{x}_t$ is a vector of past realization of x, α_i, β_j are the parameters characterizing the possible alternative demand and supply functions, and G is the excess demand function. This model, implicitly defines the dynamic evolution of the state variable x, the weights $\frac{\phi_{i,j}}{IJ}$ being both time dependent and not summing up to one.

In conclusion, it would be interesting to extend the model we have studied to include the case with endogenous weights, and thus reconnect with the vast literature on learning.

5 Concluding remarks

In this work we have studied, for a fairly general class of agent-based discrete-time dynamic economic models, the interdependence between the level of agents' heterogeneity and local stability conditions of the implied dynamics under two different, yet complementary, points of view.

First, we have shown that the common *reductionist* assumption of equivalence between a natively heterogeneous model and the corresponding model with a representative agent is granted under the set of assumptions in Section 2 - mainly the absence of inertia in expectations, coupled with linearly convex aggregation of heterogeneity. Once inertia affects agent's behavior, this result no more holds and the task of describing the dynamics becomes increasingly difficult - due to the surge in the dimensionality of the system resulting from increasing heterogeneity. For this case, in a set of different scenarios, we have shown how stability conditions can be determined explicitly, irrespective of the number of distinct agent types - hence irrespective of the dimensionality of the system.

Second, we have shown that an asymptotic result regarding the probability of stability holds under fairly general conditions on the distribution of types within the whole population. Indeed, the probability of stability of the equilibrium of an economy, as could be estimated by an outside observer (typically a policy maker), will tend to 0 or 1 depending on the value of a set of behavioral and structural parameters. We refer to this fact with the term *polarization of probabilities*. We have highlighted in the various examples the scope for manoeuvre for the policy maker in the sense of avoiding booms and busts and favouring convergence towards steady states.

Various dimensions in which this work might be extended have been presented that include agents that respond qualitatively in opposite directions to changes in the state variable (positive / negative feedback), alternative ways of aggregating heterogeneity, and endogenous (rather than constant) weights of the agents and learning.

Appendix

We need the following notation. For $n \in \mathbb{N}$, define $\mathbb{F}_n^\infty := \sigma(\alpha_n, \alpha_{n+1}, \ldots)$ and let $\mathcal{H} := \bigcap_{m=1}^\infty \mathbb{F}_m^\infty$ be the tail σ-algebra generated by the α_i.

Proof (of Proposition 2**).** *(i)* follows from the Kolmogorov's Zero-One law, observing that $\phi_i g_x(x^*, \alpha_i)$ are independent random variables and that $C \in \mathcal{H}$. Note that this holds even if $\phi_i g_x(x^*, \alpha_i) \notin L^1(\Omega, \mathbb{F}, \mathbb{P})$. *(ii)* is a straightforward application of the well-known Two-Series Theorem (see Theorem 2 in [43], IV.§2). *(iii)*: let $X_i = g_x(x^*, \alpha_i)$ have mean μ and variance σ^2, and $Z = \sum_i^n \phi_i X_i - \mu$, so that $\mathbb{E}(Z) = 0$, $\mathbb{V}(Z) = \sigma^2 \sum \phi_i^2$. Using Chebyshev's inequality we get

$$\mathbb{P}(|Z| < \epsilon) \geq 1 - \frac{\sigma^2 \sum \phi_i^2}{\epsilon^2}$$

which, as $n \to \infty$, in turn implies (8). Notice that *(iii)* follows immediately from the Strong Law of Large Numbers (SLLN) in the special case $\phi_i = 1/n$.

■

Proof (of the almost sure convergence in Example 16**).** For $i \leq n$, take independent $\beta_i \sim L(b_i)$ and fixed weights ϕ_i. Define $a_i := \sqrt{\frac{b_i}{\phi_i}}$ and $\zeta_i := \phi_i \beta_i$. Note that $\zeta_i \sim L(a_i^2)$ and in our Example 16 we assumed that $\sum_{i=1}^n a_i$ converges as n tends to infinity. Now consider the upper truncation of β_i at the threshold $c > 0$, i.e. set $\zeta_i^c := \zeta_i I_{\{\zeta_i \leq c\}}$. To prove the almost sure convergence of $\sum_{i=1}^n \zeta_i$, we will use the following well-known result.

Kolmogorov's Three-Series Theorem (see [43], IV.§2) Let $\zeta_1, \zeta_2, \ldots$ be a sequence of independent random variables. A necessary condition for the convergence of $\sum_{i=1}^n \zeta_i$ with probability 1 is that the series

$$\sum_{i=1}^n \mathbb{E}[\zeta_i^c], \qquad \sum_{i=1}^n \mathbb{V}[\zeta_i^c], \qquad \sum_{i=1}^n \mathbb{P}(|\zeta_i| \geq c) \tag{52}$$

converge for every $c > 0$. A sufficient condition is that these series converge for some $c > 0$.

Thanks to the Three-Series theorem, we fix $c = 1$ and we show the convergence of the three series. First, observe that we have $\lim_i a_i \to 0$. Therefore, there exists $\bar{n} \in \mathbb{N}$ such that for $i \geq \bar{n}$, it holds

$$\frac{1}{\sqrt{2\pi}[1 - N(a_i)]} \leq M < +\infty, \tag{53}$$

for some $M > 0$. Using (53) and the fact that for a Levy distributed variable, density and cumulative probability functions are

$$f(x) = \sqrt{\frac{c}{2\pi}} \frac{e^{-c/2x}}{x^{3/2}}, \qquad F(x) = 2\left(1 - N\left(\sqrt{\frac{c}{x}}\right)\right)$$

we find

$$\begin{aligned}
\lim_n \sum_{i=\bar{n}+1}^n \mathbb{E}[\zeta_i^1] &= \lim_n \sum_{i=\bar{n}+1}^n \left(\frac{a_i e^{-a_i^2/2}}{\sqrt{2\pi}[1 - N(a_i)]} - a_i^2 \right) \\
&\leq M \lim_n \sum_{i=\bar{n}+1}^n \left(a_i e^{-a_i^2/2} - a_i^2 \right) \\
&\leq M \lim_n \sum_{i=\bar{n}+1}^n a_i < +\infty
\end{aligned}$$

thanks to our assumption. For the second series, we have

$$\begin{aligned}
\lim_n \sum_{i=\bar{n}+1}^n \mathbb{V}[\zeta_i^1] &\leq \lim_n \sum_{i=\bar{n}+1}^n \mathbb{E}[(\zeta_i^1)^2] \\
&= \frac{1}{3} \lim_n \sum_{i=\bar{n}+1}^n \left(a_i^4 + \frac{a_i e^{-a_i^2/2}}{\sqrt{2\pi}[1 - N(a_8 i)]} - \frac{a_i^3 e^{-a_i^2/2}}{\sqrt{2\pi}[1 - N(a_i)]} \right) \\
&\leq \frac{1}{3} \lim_n \sum_{i=\bar{n}+1}^n \left(a_i^4 + M a_i \right) < +\infty,
\end{aligned}$$

since $a_i^4 < a_i$ definitely. Finally, for the third series it is sufficient to note that

$$\begin{aligned}\mathbb{P}(\zeta_i \geq 1) = 2N(a_i) - 1 &= \frac{2}{\sqrt{2\pi}} \int_0^{a_i} e^{-z^2/2} dz \\ &\leq \frac{2}{\sqrt{2\pi}} a_i\end{aligned}$$

holds for every i. ■

Define henceforth

$$\eta_i = \phi_i g_x(x^*, y^*, \alpha_i) \text{ and } \xi_i = g_y(x^*, y^*, \alpha_i)$$

.

Proof (of Lemma 7)**.**
(i) readily follows from Schur's complement formula. To prove *(ii)*, suppose all ξ_i are distinct and non-zero (these last cases can be treated similarly). Without loss of generality, we set $\xi_1 < \xi_2 < \ldots < \xi_n$. Using the formula of p_n, we see that

$$sign(p_n(\xi_i)p_n(\xi_{i+1})) = sign(\xi_i \xi_{i+1} f_y(y^*)^2 \eta_i \eta_{i+1} (-1)^{2i-1}). \tag{54}$$

Hence, if ξ_i and ξ_{i+1} have the same sign, then p_n changes its sign in the interval (ξ_i, ξ_{i+1}). Now we distinguish two main cases. First, if all ξ_i have the same sign, then p_n has $(n-1)$ real roots, one in every interval (ξ_i, ξ_{i+1}). Moreover, zero is always a root of p_n. Hence the only remaining root must be real. On the contrary, if all ξ_i do not have the same sign, then there exists a unique h such that $\xi_h < 0 < \xi_{h+1}$ and $p_n(\xi_h)p_n(\xi_{h+1}) > 0$. Now, observe that

$$sign(p_n(\xi_1)) = sign(f_y(y^*)\eta_1), \quad sign(p_n(\xi_n)) = (-1)^n sign(f_y(y^*)\eta_n). \tag{55}$$

Besides, we have $\lim_{\lambda \to -\infty} p_n(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} p_n(\lambda) = (-1)^{n+1}\infty$. Therefore, considering the intervals (ξ_i, ξ_{i+1}) and the behavior at infinity, p_n changes its sign exactly $n-1$ times (there are 4 cases, depending on n even or odd and $sign(p_n(\xi_1)) = \pm 1$; they can be analyzed separately). Again, zero is always a root of p_n and the remaining root must be real. Finally, *(iii)* is a direct consequence of the previous argument. We also note that if $f_y(y^*)\eta_1 > 0$, then an eigenvalue is greater than $\max_i\{\xi_i\}$. Otherwise, if $f_y(y^*)\eta_1 < 0$, then an eigenvalue is smaller than $\min_i\{\xi_i\}$. ■

Proof (of Proposition 8)**.** The event $\bar{A} := \limsup_i A_i \equiv \{\omega \in \Omega : |\xi_i(\omega)| > 1 \text{ infinitely often}\}$ belongs to the tail σ-algebra $\mathcal{H}$. Therefore, $\mathbb{P}(\bar{A})$ takes only the values 0 or 1. From Borel-Cantelli Lemma, we have $\mathbb{P}(\bar{A}) = 0$ if and only if $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$. Therefore, our hypothesis implies $\mathbb{P}(\bar{A}) = 1$. In other words, when n tends to infinity, at least one eigenvalue of J_n lies outside the interval $(-1, 1)$ almost surely. ■

Proof (of Proposition 9)**.**
(i) We prove the result when $\xi_i \in (0, 1)$, the other case being similar. Using the formula for p_n, we see that

$$p_n(\xi_1) < 0, \qquad sign(p_n(\xi_n)) = (-1)^n. \tag{56}$$

Hence, if n is even (odd) it follows $p_n(\xi_n) \underset{(>0)}{<} 0$ and $\lim_{\lambda \to +\infty} p_n(\lambda) = \underset{(+\infty)}{-\infty}$. Therefore, we have

$$\underset{(n \text{ odd})}{n \text{ even}} \Rightarrow \rho(J_n) < 1 \iff p_n(1) \underset{(>0)}{<} 0 \iff 1 + f_y(y^*) \sum_{i=1}^{n} \frac{\eta_i}{\xi_i - 1} > 0. \tag{57}$$

Next, observe that, using the same argument of the proof of Proposition 2 (iii), $\sum_{i=1}^{n} \frac{\eta_i}{\xi_i - 1}$ converges almost surely to μ as n grows to infinity, so (34) thus follows (even when $\mu = -\infty$).
(ii) It is similar to (i) and therefore omitted. ■

In order to prove Proposition 12 we need the following Lemma, which is the counterpart of Lemma 7.

Lemma 18 *Let $p_n(\lambda)$ be the characteristic polynomial of the Jacobian of* (38)*. Assume $\eta_i) \in (-1, 1)$ all have the same sign, all η_i have the same sign and $f_y(x^*, y^*)\eta_1\xi_1 > 0$. Then:*

(i) $p_n(\lambda) = \prod_{i=1}^{n}(\xi_i - \lambda)(f_x(x^*, y^*) - \lambda) - \lambda f_y(x^*, y^*) \sum_{i=1}^{n} \eta_i \prod_{j \neq i}(\xi_j - \lambda)$.

(ii) p_n *has* $(n+1)$ *real roots.*

(iii) Exactly $n-1$ roots of p_n belong to the interval $[\min_i\{\xi_i\}, \max_i\{\xi_i\}]$.

Proof. *(i)* p_n is obtained via Schur's formula. *(ii)-(iii)* Inspecting the sign of p_n, we see that

$$\begin{aligned} sign(p_n(\xi_i)p_n(\xi_{i+1})) &= sign(-\xi_i\xi_{i+1}\eta_i\eta_{i+1}) \\ sign(p_n(\xi_1)) &= sign(-f_y(x^*, y^*)\xi_1\eta_1) \\ sign(p_n(\xi_n)) &= sign((-1)^n f_y(x^*, y^*)\xi_n\eta_n) \end{aligned}$$

Hence, if ξ_i have the same sign we are able to locate at least $n-1$ roots of p_n in the interval $[\min_i\{\xi_i\}, \max_i\{\xi_i\}]$. Besides, if $f_y(x^*, y^*)\xi_1\eta_1 > 0$, then we see that another root of p_n lies in $(-\infty, \min_i\{\xi_i\})$, since $sign(p_n(\xi_1)) = -1$ and $\lim_{\lambda\to-\infty} p_n(\lambda) = +\infty$. Finally, distinguishing the cases n even or odd shows that the only remaining root of p_n lies in $(\max_i\{\xi_i\}, +\infty)$ thanks to the intermediate value theorem. ■

Proof (of Proposition 12**).** The proof proceeds in the same way as in Proposition 9, observing that stability requires at the same time $p_n(-1) > 0$ and $p_n(1) < 0 (> 0)$ if n is even (odd). ■

References

[1] Agliari, A., Massaro, D., Pecora, N., Spelta, A., 2017. Inflation targeting, recursive inattentiveness and heterogeneous beliefs. *Journal on Money, Credit & Banking* 49/7, 1587-1619.

[2] Anufriev, M., Assenza, T., Hommes, C., Massaro, D., 2013. Interest rate rules and macroeconomic stability under heterogeneous expectations. *Macroeconomic Dynamics* 17/8, 1574-1604.

[3] Anufriev, M., Hommes, C., Makarewicz, T., 2019. Simple forecasting heuristics that make us smart: evidence from different market experiments *Journal of the European Economic Association* 17/5, 1538-1584.

[4] Assenza, T., Bao, T., Hommes, C., Massaro, D., 2014. Experiments on expectations in macroeconomics and finance. In: John Duffy (ed.), Research in Experimental Economics: Experiments in Macroeconomics, volume 17, pages 11-70, Emerald Publishing Ltd.

[5] Agiza, H.N., Elsadany, A.A., 2003. Nonlinear dynamics in the Cournot duopoly game with heterogeneous players. *Physica A: Statistical Mechanics and its Applications*, 320, 512-524.8

[6] Bao, T., Hommes, C., Sonnemans, J. Tuinstra, J., 2012. Individual expectations, limited rationality and aggregate outcomes. *Journal of Economic Dynamics & Control* 36/8, 1101-1120.

[7] Baur, D.G., Glover, K.J., 2014. Heterogeneous expectations in the gold market: Specification and estimation. *Journal of Economic Dynamics & Control* 40, 116-133.

[8] Berck, P., Perloff, J.M., 1998. The dynamic annihilation of a rational competitive fringe by a low-cost dominant firm. *Journal of Economic Dynamics & Control* 12/4, 659-678,

[9] Bonacina, M., Gullì. F., 2007. Electricity pricing under "carbon emissions trading": A dominant firm with competitive fringe model. *Energy Policy* 35/8, 4200-4220.

[10] Boswijk, H.P., Hommes, C.H., Manzan, S., 2007. Behavioral heterogeneity in stock prices. *Journal of Economic Dynamics & Control* 31, 1938-1970.

[11] Brock, W.A., Hommes, C.H., 1997. A rational route to randomness. *Econometrica* 65, 1059-1095.

[12] Brock, W.A., Hommes, C.H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control* 22, 1235–1274.

[13] Brock, W.A., Hommes, C.H., Wagener, F.O.O., 2005. Evolutionary dynamics in markets with many trader types. *Journal of mathematical Economics* 41, 7-42.

[14] Brock, W.A., Hommes, C.H., Wagener, F.O.O., 2009. More hedging instruments may destabilize markets. *Journal of Economic Dynamics & Control* 33, 1912-1928.

[15] Chavas, J.P., 2000. On information and market dynamics: the case of the U.S. beef market. *Journal of Economic Dynamics & Control* 24, 833-853.

[16] Chen, S.H., Chang, C.L., Du, Y.R., 2012. Agent-based economic models and econometrics. *Knowledge Engineering Review* 27/2, 187-219.

[17] Colander, D., Howitt P., Kirman, A., Leijonhufvud A., Mehrling, P., 2008. Beyond DSGE Models: Toward an Empirically Based Macroeconomics. *American Economic Review: Papers & Proceedings*, 98/2, 236–240.

[18] Colucci, D., Valori, V., 2006. Ways of learning in a simple economic setting: A comparison. *Chaos, Solitons & Fractals* 29/3, 653-670.

[19] Colucci, D., Valori, V., 2008. Asset price dynamics when behavioural heterogeneity varies. *Computational Economics* 32, 3-20.

[20] Colucci, D., Valori, V., 2011. Adaptive expectations and cobweb phenomena: Does heterogeneity matter? *Journal of Economic Dynamics & Control* 35, 1307-1321.

[21] Colucci, D., Valori, V., 2015. Stabilizing inflation in a simple monetary policy model with heterogeneous agents. *Mathematics and Computers in Simulation* 108, 233-244.

[22] Cornea-Madeira, A., Hommes, C.H., Massaro, D., 2019. Behavioral Heterogeneity in U.S. Inflation Dynamics. *Journal of Business & Economic Statistics* 37/2, 288-300.

[23] Day, R. H., Huang, W., 1990. Bulls, bears and market sheep. *Journal of Economic Behavior and Organization* 14, 299-329.

[24] Diks, C.G.H., van der Weide, R., 2005. Herding, a-synchronous updating and heterogeneity in memory in a CBS. *Journal of Economic Dynamics and Control* 29, 741-763.

[25] Fleurbaey, M., 2009. Beyond GDP: The Quest for a Measure of Social Welfare. *Journal of Economic Literature* 47/4, 1029-1075.

[26] Föllmer, H., Horst, U., Kirman, A., 2005. Equilibria in financial markets with heterogeneous agents: a probabilistic perspective. *Journal of Mathematical Economics* 41, 123-155.

[27] Forni, M., Lippi, M., 1997. Aggregation and the Microfoundation of Dynamic Macroeconomics. Clarendon Press.

[28] Gallegati, M., Palestrini, D., Delli Gatti, D., Scalas, E., 2006. Aggregation of heterogeneous interacting agents: the variant representative agent framework. *Journal of Economic Interaction and Coordination* 1/1, 5-19.

[29] Grandmont, J-M. 1998. Expectations formation and stability of large socioeconomic systems. *Econometrica* 66/4, 741-781.

[30] Harrison, G.W., Rutström, E.E. 2009. Expected utility theory and prospect theory: one wedding and a decent funeral. *Experimental Economics* 12, 133-158.

[31] Hartley, J., 1997. The representative agent in macroeconomics. Routledge.

[32] Haruvy, E., Lahav, Y., Noussair, C.N., 2007. Traders' Expectations in asset markets: experimental evidence. *The American Economic Review* 97, 1901-1920.

[33] Heemeijer, P., Hommes, C.H., Sonnemans, J., Tuinstra, J., 2009. Price stability and volatility in markets with positive and negative expectations feedback: An experimental investigation. *Journal of Economic Dynamics & Control* 35, 1052-1072.

[34] Hommes, C.H., 1994. Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand. *Journal of Economic Behavior and Organization* 24/3, 315-335.

[35] Hommes, C.H., 2011. The heterogeneous expectations hypothesis: Some evidence from the lab. *Journal of Economic Dynamics & Control* 35, 1-24.

[36] Hommes, C.H., Sonnemans, J., Tuinstra, J., van de Velden, H., 2005. Coordination of expectations in asset pricing experiments *Review of Financial Studies* 18/3, 955-980.

[37] Kirman, A., 1992. Whom or what does the representative individual represent? *Journal of Economic Perspectives* 6/2, 117–136.

[38] McFadden, D., 1973. Conditional logit analysis of qualitative choice behavior. In Frontiers in Econometrics, New York (ed. P. Zarembka), pp. 105–142. Academic Press, NY.

[39] Onozaki, T., Sieg, G. Yokoo, M., 2000. Complex dynamics in a cobweb model with adaptive production adjustment. *Journal of Economic Behavior and Organization* 41, 101-115.

[40] Pecora, N., Spelta, A., 2017. Managing monetary policy in a New Keynesian model with many beliefs types. *Economics Letters* 150, 53-58.

[41] Reitz, S., Westerhoff, F.H., 2007. Commodity price cycles and heterogeneous speculators: a STAR-GARCH model. *Empirical Economics* 33, 231–244.

[42] Schmitt, N., Westerhoff, F., 2019. Heterogeneity, spontaneous coordination and extreme events within large-scale and small-scale agent-based financial market models. *Journal of Evolutionary Economics* 27/5, 1041-1070.

[43] Shiryaev, A.N., 1996. Probability, second ed., Springer-Verlag.

[44] Sims, C.A., 1980. Macroeconomics and Reality. *Econometrica*, 48, 1-48.

[45] Stiglitz, J. E., Gallegati, M., 2011. Heterogeneous interacting agent models for understanding monetary economies. *Eastern Economic Journal*, 37, 6–12.

[46] Stoker, T., 1993. Empirical approaches to the problem of aggregation over individuals. *Journal of Economic Literature* 314, 1827–1874.

[47] von Gaudecker, H.-M., Wogrolly, A., 2021. Heterogeneity in households' stock market beliefs: Levels, dynamics, and epistemic uncertainty. *Journal of Econometrics*, In Press.