Anonymous, neutral and reversal symmetric majority rules

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Abstract

In the standard arrovian framework and under the assumptions that individual preferences and social outcomes are linear orders over the set of alternatives, we provide necessary and sufficient conditions for the existence of anonymous, neutral and reversal symmetric rules and for the existence of anonymous, neutral, reversal symmetric majority rules.

Keywords: Social welfare function; anonymity; neutrality; reversal symmetry; majority; linear order; group theory.

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1 Introduction

There are many procedures that members of a committee can conceive to aggregate their preferences over a given set of alternatives into a strict ranking of these alternatives. In the present paper, we deal with the problem of finding out conditions under which it is possible to design aggregation procedures satisfying the well-known principles of anonymity, neutrality, majority and reversal symmetry. Those principles are usually considered able to guarantee some extent of equity and fairness of the collective decisions and are often invoked in social choice theory.

We study the existence of such procedures following the approach based on group theory developed in Bubboloni and Gori (2013). We also work in the same setting and notation of that paper. More precisely, we consider $h \geq 2$ individuals and $n \geq 2$ alternatives to be ranked, and we assume that individual preferences and social preferences (or decision outcomes) are strict rankings over the set of alternatives. A preference profile is a list of h strict rankings each of them associated with the name of a specific individual and representing her preferences. We call rule any function from the set of preference profiles to the set of social preferences: a rule represents then a particular decision procedure allowing to determine a strict ranking of alternatives from any conceivable individual preferences expressed by individuals.

A rule is anonymous if it associates the same social preference with pairs of preference profiles such that we can get one from the other by figuring to permute individual names. A rule is instead neutral if, for every pair of preference profiles such that we can get one from the other by figuring to permute alternative names, the social preferences associated with them coincide up to the considered permutation of the names. It is quite simple to show that the arithmetical condition[1]

$$\gcd(h, n!) = 1, \tag{1}$$

first introduced by Moulin (1983, Theorem 1, p.25) as a necessary and sufficient condition for the existence of anonymous, neutral and efficient social choice functions[2], is necessary for the existence of anonymous and neutral rules. Bubboloni and Gori (2013, Theorem 6) proved that condition (1) is also sufficient for the existence of anonymous and neutral rules.

Given an integer ν not exceeding the number h of individuals but exceeding half of it, a ν-majority rule is a rule associating with every preference profile a social preference ranking an alternative over another one if, according to the considered preference profile, it is preferred to the other by at least ν individuals. A minimal majority rule is instead a rule having the property that, for every preference profile, the corresponding social preference is consistent with all majority thresholds not generating Condorcet-cycles for that profile.

It is known[3] that ν-majority rules exist if and only if h, n and ν satisfy the inequality

$$\nu > \frac{n-1}{n} h \tag{2}$$

that Greenberg (1979) realized to be sufficient to guarantee the existence of social choice functions satisfying the ν-majority principle. Bubboloni and Gori (2013, Theorems 16 and 13) also proved that condition (1) is sufficient for the existence of anonymous and neutral minimal majority rules and that conditions (1) and (2) are sufficient for the existence of anonymous and neutral ν-majority rules[4].

In this paper we want to focus on the property of reversal symmetry, as well. Given a preference, its reversal is the preference whose best alternative is the worst alternative of the given preference, whose second best alternative is the second worst alternative of the given preference, and so on. A rule is reversal symmetric if, for any pair of preference profiles such that one is obtained by the other reversing each individual preference, it associates with them social outcomes that are one the reversed of the other. In other words, if everybody in the society completely changes her mind about how much she likes alternatives, then that implies a complete change in social values. The purpose of our paper is to show that each of the previously mentioned existence results by Bubboloni and Gori (2013) can be generalized involving reversal symmetry too. Indeed, we are able to prove the theorems below[5].

Theorem A. *If* (1) *holds true, then there are at least*[6]

$$\left(2^{\lfloor \frac{n}{2} \rfloor} \left\lfloor \frac{n}{2} \right\rfloor !\right)^{\left\lceil \frac{(h+n!-1)!}{2(n!-1)!n!h!} \right\rceil}$$

anonymous, neutral and reversal symmetric rules.

Theorem B. *If* (1) *holds true, then there exists an anonymous, neutral and reversal symmetric minimal majority rule.*

[1] With gcd we denote the greatest common divisor.

[2] A social choice function is a function from the set of preference profiles to the set of alternatives.

[3] See Can and Storken (2012, Example 4) and Bubboloni and Gori (2013, Theorem 10).

[4] We refer to Bubboloni and Gori (2013) for further details and comments about the literature related to the principles of anonymity, neutrality and majority.

[5] Observe Theorems A, B and C are just a rephrasing of Theorems 2, 3 and 4, respectively.

[6] Given $x \in \mathbb{R}$, $\lceil x \rceil$ denotes its superior integer part and $\lfloor x \rfloor$ denotes its inferior integer part.

Theorem C. *If* (1) *and* (2) *hold true, then there exists an anonymous, neutral and reversal symmetric ν-majority rule.*

As already emphasized, the algebraic approach we follow is crucial for proving the described results. Moreover, as a byproduct, those techniques allow also to get concrete methods to build all the rules of any considered type and formulas to exactly count them once h, n are given (see Propositions 12, 15 and 18, taking into account also Proposition 13). The last part of the paper is devote to show how those methods and formulas can be actually applied when $n = 3$ and $h = 5$.

2 Mathematical tools and the model

2.1 Arithmetics

Along the paper the symbol $\geq$ denotes the usual linear order in $\mathbb{R}$. If X is a finite set, we denote with $|X|$ its order. Given $m, n \in \mathbb{N}$, we will use the notation $m \mid n$ to say that m divides n. Given $x \in \mathbb{R}$, we denote with $\lceil x \rceil$ its superior integer part, that is the minimum $z \in \mathbb{Z}$ with $z \geq x$ and with $\lfloor x \rfloor$ its inferior integer part, that is the maximum $z \in \mathbb{Z}$ with $z \leq x$. Given $r \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r$, we denote the greatest common divisor and the least common multiple of $\lambda_1, \ldots, \lambda_r$ by $\gcd(\lambda)$ and $\operatorname{lcm}(\lambda)$, respectively. Given $k \in \mathbb{N}$, we define the set

$$\Pi(k) = \bigcup_{r=1}^{k} \left\{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r : \sum_{j=1}^{r} \lambda_j = k, \lambda_1 \geq \ldots \geq \lambda_r \right\}$$

whose elements are called *partitions* of k. In other words, a partition of k is a decreasing list of positive integers whose sum is k. If $\lambda \in \Pi(k)$, we write $\lambda \vdash k$ and we call each component of λ a *part* of λ.

2.2 Groups

Let G be a finite group with operation denoted by juxtaposition, neutral element 1 and inverse of $g \in G$ denoted by g^{-1}. Two elements $g_1, g_2 \in G$ *commute* if $g_1 g_2 = g_2 g_1$. A group G is *abelian* if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$. A non-empty subset U of G is called a subgroup if for all $u_1, u_2 \in U$ we have $u_1 u_2$, $u_1^{-1} \in U$. If U is a subgroup of G we write $U \leq G$. Clearly $\{1\}$, $G \leq G$. If $U \leq G$ and $U \neq G$ we say that U is a *proper* subgroup of G and we write $U < G$. Since G is finite, given $\varnothing \neq U \subseteq G$, we have that $U \leq G$ if and only if for all $u_1, u_2 \in U$ we have $u_1 u_2 \in U$. If $U \leq G$, the *index* of U in G is defined as $[G : U] = \frac{|G|}{|U|}$ and, by Lagrange Theorem, both $|U|$ and $[G : U]$ are natural numbers dividing $|G|$. Moreover, if $U \leq V \leq G$ then

$$[G : U] = [G : V][V : U]. \tag{3}$$

If $Y \subseteq G$, the subgroup generated by Y denoted by $\langle Y \rangle$, is defined by the intersection of all the subgroups of G containing Y. Observe that $\langle Y \rangle$ is the smallest subgroup of G containing Y. When $Y = \{g\}$, for some $g \in G$, then $\langle g \rangle$ coincide with the finite set of its integer (or natural) powers, that is, $\langle g \rangle = \{g^j : j \in \mathbb{Z}\}$. The natural number $|\langle g \rangle|$, briefly denoted by $|g|$, is called the *order* of g. If s is the minimum exponent in $\mathbb{N}$ such that $g^s = 1$, then $\langle g \rangle = \{g^j : j \in \{1, \ldots, s\}\}$ and $|g| = s$. If $s' \in \mathbb{Z}$ is such that $g^{s'} = 1$, then $|g| \mid s'$. If $G = \langle g \rangle$ we say that G is a cyclic group. Clearly every cyclic group is abelian.

For $X, Y \subseteq G$ we can consider their *product* $XY = \{xy \in G : x \in X, y \in Y\} \subseteq G$. The product of subsets is associative, that is if $X, Y, Z \subseteq G$, then $X(YZ) = (XY)Z$. When $X = \{x\}$ we write, for brevity xY instead of $\{x\}Y$.

We are expecially interested to the case in which one of the sets is a singleton and to the case in which both the subsets are subgroups. If $X = \{x\}$, then $|XY| = |Y|$ because the map $\alpha : Y \to XY$ defined as $\alpha(y) = xy$ is a bijection. If $X, Y \leq G$ the subset XY of G is not, in general, a subgroup. Anyway its order is well known and given by

$$|XY| = \frac{|X||Y|}{|X \cap Y|}. \tag{4}$$

A subgroup U of G is called *normal* if for all $g \in G$ we have $gU = Ug$. When U is normal in G we use the notation $U \trianglelefteq G$. The normal subgroups play a central role in group theory but in this paper we will need very few properties about them. For instance the fact that if $U \trianglelefteq G$ and $V \leq G$ then $UV = VU \leq G$ coincides with the subgroup $\langle U \cup V \rangle$ generated by U and V. It is well known that if $U \leq G$ and $[G : U] = 2$ then $U \trianglelefteq G$. Moreover if G is abelian then each subgroup of G is normal.

If $Y \subseteq G$ the *centralizer* of Y in G is defined as

$$C_G(Y) = \{g \in G : yg = gy, \forall y \in Y\}$$

and it is immediately checked that $C_G(Y) \leq G$.

2.3 Symmetric groups

All the groups in this paper are finite permutation groups. We give here the basic notation and terminology required for them.

First of all if X_1, X_2, X_3 are three finite sets and $f_1 : X_1 \to X_3$, $f_2 : X_2 \to X_1$ are functions, we denote the (right-to-left) composition of f_1 and f_2 by their juxtaposition f_1f_2, so that $f_1f_2 : X_2 \to X_3$ and $f_1f_2(x) = f_1(f_2(x))$ for all $x \in X_2$. Let X be a finite set and $\mathfrak{F}(X)$ the set of functions from X to X. Given $f_1, f_2 \in \mathfrak{F}(X)$ we call $f_1f_2 \in \mathfrak{F}(X)$ the *product* between f_1 and f_2. When $X \neq \varnothing$, the subset $\mathrm{Sym}(X)$ of $\mathfrak{F}(X)$ made up by the bijective functions is a finite group with respect to the product of functions and it is called the symmetric group on X. The neutral element of $\mathrm{Sym}(X)$ is the identity function, denoted by id_X or simply by id. If $|X| = 1$ then $Sym(X) = \{id\}$ is the *trivial group*.

Fix $k \in \mathbb{N}$ and let $K = \{1, \dots, k\}$. We denote $\mathrm{Sym}(K)$ simply by S_k and call its elements permutations on k objects. It is well known that $|S_k| = k!$. We say that $\gamma \in S_k$ is a *cycle of length* 1 if $\gamma = id$. We say that $\gamma \in S_k$ is a *cycle of length* $l \geq 2$ if there exist distinct $x_1, \dots, x_l \in K$ such that, for every $j \in \{1, \dots, l-1\}$, $\gamma(x_j) = x_{j+1}$ and $\gamma(x_l) = x_1$, while $\gamma(x) = x$ for any $x \in K \setminus \{x_1, \dots, x_l\}$. In that case we write $\gamma = (x_1 \dots x_l)$. For instance, if $k \geq 3$, (123) is the cycle of length 3 which maps 1 into 2, 2 into 3, 3 into 1 and any further element (if any) into itself. We say that a cycle is *proper* if its length is greater than 1. Note that if γ is a cycle of length l, then $|\gamma| = l$. Two cycles γ_1, γ_2 are *disjoint* if one of them is id or if $\gamma_1 = (x_1 \dots x_{l_1})$ and $\gamma_2 = (y_1 \dots y_{l_2})$ with suitable $l_1, l_2 \geq 2$ and $x_1, \dots, x_{l_1}, y_1, \dots, y_{l_2} \in K$ are such that $\{x_1, \dots, x_{l_1}\} \cap \{y_1, \dots, y_{l_2}\} = \varnothing$. Disjoint cycles commute, that is, making their product in any order you get the same permutation. Recall that the a cycle has order l if and only if has length l.

Let $\sigma \in S_k$. Then $|\sigma|$ divides $k!$ and σ splits uniquely, up to the order, into the product $\sigma = \gamma_1 \cdots \gamma_r$ of $r \geq 1$ pairwise disjoint cycles $\gamma_1, \dots, \gamma_r \in S_k$ with $|\gamma_1| + \cdots + |\gamma_r| = k$. We say that $x \in K$ is a *fix point* for $\sigma \in S_k$ if $\sigma(x) = x$. Note that σ has as many fix points as the number of $\gamma_j = id$ in its split. Given $\lambda = (\lambda_1, \dots, \lambda_r) \vdash k$, we say that $\sigma \in S_k$ is of *type* λ if $\sigma = \gamma_1 \cdots \gamma_r$ for some $\gamma_1, \dots, \gamma_r$ pairwise disjoint cycles and, for every $j \in \{1, \dots, r\}$, $|\gamma_j| = \lambda_j$. Note that σ^{λ_j} has at least λ_j fix points and that $|\sigma| = \mathrm{lcm}(\lambda)$. The number of permutations of the same type λ is given by the index in S_k of the centralizer of a permutation of type λ. Given the cycle $\gamma = (x_1 \dots x_l)$ and $\sigma \in S_k$, then $\sigma\gamma\sigma^{-1} = (\sigma(x_1) \dots \sigma(x_l))$ is itself a cycle. For instance $(123)(124)(132) = (234)$.

Given $a \in \mathbb{N}$ and $(k_1, \ldots, k_a) \in \mathbb{N}^a$, the direct product $\times_{j=1}^a S_{k_j}$ of the groups S_{k_j} is itself a group with the usual component by component operation. For $b \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, the notation S_k^b is used to denote the direct product of b copies of S_k if $b \geq 1$, while S_k^0 stands for the trivial group.

Any other notation and basic results used for permutations are standard (see, for instance, Wielandt (1964) and Rose (1978)).

2.4 Actions of a group on a set

Let G be a finite group and X a non-empty finite set. If $f : G \to \mathrm{Sym}(X)$ is an homomorphism[7], we say that f is an *action* of G on X or that G *acts* on X via f.

Let G acts on X via f. If U is a group and there exists an homomorphism $\iota : U \to G$, then U acts on X via $f\iota$. If ι is injective we identify the action $f\iota$ with the restriction of f to $\iota(U)$. In particular if $U \leq G$, and G acts on X then also U acts on X, because the inclusion ι of U into G given by $\iota(u) = u$ for all $u \in U$, is obviously an injective homomorphism.

Given $x \in X$ and $g \in G$, we say that the action of g on x is given by $f(g)(x) \in X$. When we do not need an explicit reference to f we prefer to write x^g instead of $f(g)(x)$. Note that, within that notation, the fact that f is an action means that for all $g_1, g_2 \in G$ and all $x \in X$, we have $(x^{g_1})^{g_2} = x^{g_2 g_1}$.

For every $x \in X$, the subset of X given by $x^G = \{x^g \in X : g \in G\}$ is called the G-*orbit* of x and is denoted by x^G. Each G-orbit is nonempty and finite and it is well known that, given $x, y \in X$, $x^G \cap y^G = \varnothing$ or $x^G = y^G$. Moreover $\bigcup_{x \in X} x^G = X$. The set of orbits $\mathcal{O}(G) = \{x^G : x \in X\}$ is nonempty and finite and we put $|\mathcal{O}(G)| = R(G)$. If $U \leq G$ and $x \in X$, then $x^U \subseteq x^G$ and $R(U) \geq R(G)$; in other words restricting an action, the orbit size does not increase while the number of orbits does not decrease. Any vector $(x^j)_{j=1}^{R(G)} \in X^{R(G)}$ such that $\{x^{j\,G} : j \in \{1, \ldots, R(G)\}\} = \mathcal{O}(G)$, is called *a system of representatives* of the G-orbits. The set of the systems of representatives of the G-orbits is nonempty and denoted by $\mathfrak{S}(G)$. If $(x^j)_{j=1}^{R(G)} \in \mathfrak{S}(G)$, then $\{x^{j\,G} : j \in \{1, \ldots, R(G)\}\}$ is a partition of X.[8]

For every $x \in X$ and $U \leq G$ the *stabilizer* of x in U is the subgroup of U defined by

$$\mathrm{Stab}_U(x) = \{u \in U : x^u = x\}.$$

Note that $\mathrm{Stab}_U(x) = U \cap \mathrm{Stab}_G(x)$. It is well known that the order of the G-orbit x^G can be expressed in terms of the stabilizer of x in G by

$$|x^G| = \frac{|G|}{|\mathrm{Stab}_G(x)|}, \tag{5}$$

and, in particular, the order of each orbit divides $|G|$.

We can compare the order of the G-orbit x^G with that of the U-orbit x^U, for $U \leq G$, through the index of U in G and the index of $\mathrm{Stab}_U(x)$ in $\mathrm{Stab}_G(x)$, because

$$|x^G| = |x^U| \frac{[G : U]}{[\mathrm{Stab}_G(x) : \mathrm{Stab}_U(x)]} \leq |x^U|[G : U]. \tag{6}$$

For the purposes of this paper we need to study only the specific situation in which $[G : U] = 2$.

Lemma 1. *Let G be a group acting on the set X and $U \leq G$, with $[G : U] = 2$. Then:*

[7] Recall that if G_1 and G_2 are groups with respect to operations $*_1, *_2$, then a map $f : G_1 \to G_2$ is called an homomorphism if $f(x *_1 y) = f(x) *_2 f(y)$, for all $x, y \in G_1$. Recall also that the composition of homomorphisms is an homomorphism. A homomorphism which is a bijection is called an *isomorphism*.

[8] In the paper, a partition of a nonempty set X is a family of nonempty pairwise disjoint subsets of X whose union is X.

i) For $x \in X$, only two cases may occur:

a) $\mathrm{Stab}_G(x) = \mathrm{Stab}_U(x)$ *and* $|x^G| = 2|x^U|$, *with* x^G *given by the union of two U-orbits of equal length;*

b) $[\mathrm{Stab}_G(x) : \mathrm{Stab}_U(x)] = 2$ *and* $x^G = x^U$*. In this case, for all* $g \in \mathrm{Stab}_G(x) \setminus \mathrm{Stab}_U(x)$*, we have* $\mathrm{Stab}_G(x) = \mathrm{Stab}_U(x) < g >$.

ii) $\left\lceil \frac{R(U)}{2} \right\rceil \leq R(G) \leq R(U)$.

iii) Let $(x^i)_{i=1}^{R(U)} \in \mathfrak{S}(U)$ *and* $g \in G \setminus U$*. Then for each i with $1 \leq i \leq R(U)$, there exists an unique j with $1 \leq j \leq R(U)$ such that $(x^i)^g = (x^j)^u$ for some $u \in U$. If $j \neq i$, then $(x^i)^G = (x^i)^U \cup (x^j)^U$. If $j = i$, then $(x^i)^G = (x^i)^U$.*

Proof. i) Since $[G : U] = 2$, we have that $U \trianglelefteq G$ and thus $U \leq U\mathrm{Stab}_G(x) \leq G$. By (3) we then get $U\mathrm{Stab}_G(x) = U$ or $U\mathrm{Stab}_G(x) = G$.

a) Let $\mathrm{Stab}_G(x) \leq U$, that is $\mathrm{Stab}_G(x) = \mathrm{Stab}_U(x)$. Thus

$$|x^G| = [G : \mathrm{Stab}_G(x)] = [G : \mathrm{Stab}_U(x)] = [G : U][U : \mathrm{Stab}_U(x)] = 2|x^U|. \tag{7}$$

In particular $x^G \supset x^U$ and so there exists $g \in G \setminus U$ with $x^g \in x^G \setminus x^U$. We show that $|(x^g)^U| = |x^U|$. Being $U \trianglelefteq G$, we have

$$\begin{aligned} |(x^g)^U| &= [U : \mathrm{Stab}_U(x^g)] = [U : (g\mathrm{Stab}_G(x)g^{-1}) \cap U] = [gUg^{-1} : (g\mathrm{Stab}_G(x)g^{-1}) \cap (gUg^{-1})] \\ &= [U : \mathrm{Stab}_U(x)] = |x^U|. \end{aligned}$$

This, by (7), says that $x^G = x^U \cup (x^g)^U$ is the union of two U-orbits of equal size.

b) Let $U\mathrm{Stab}_G(x) = G$. Then, by (4), we have

$$|x^G| = [G : \mathrm{Stab}_G(x)] = [U\mathrm{Stab}_G(x) : \mathrm{Stab}_G(x)] = [U : \mathrm{Stab}_U(x)] = |x^U|$$

and also

$$[\mathrm{Stab}_G(x) : \mathrm{Stab}_U(x)] = 2, \tag{8}$$

so that $\mathrm{Stab}_U(x) \trianglelefteq \mathrm{Stab}_G(x)$.

Pick now $g \in \mathrm{Stab}_G(x) \setminus \mathrm{Stab}_U(x)$. Then we have $\mathrm{Stab}_U(x) < \mathrm{Stab}_U(x)\langle g\rangle \leq \mathrm{Stab}_G(x)$ and so, by (8) and (3), we get $\mathrm{Stab}_U(x)\langle g\rangle = \mathrm{Stab}_G(x)$.

ii) Since, by i), the G-orbits coincide with the U-orbits or are obtained gluing together two U-orbits, the result follows.

iii) Let $(x^i)_{i=1}^{R(U)} \in \mathfrak{S}(U)$ and $g \in G \setminus U$. Since $(x^i)^U$ are a partition of $\mathcal{P}$, for each $i \in \{1, \ldots, R(U)\}$, there exists $j \in \{1, \ldots, R(U)\}$ and $u \in U$, such that $(x^i)^g = (x^j)^u$. Moreover, by definition of system of representatives, j and u are uniquely determined. If $j \neq i$ then $(x^i)^G \supseteq (x^i)^U \cup (x^j)^U$ and since we know, by i), that the only two possibilities for the G-orbits are to be coincident with one U-orbit or to be the union of just two U-orbits, we deduce that $(x^i)^G = (x^i)^U \cup (x^j)^U$. if $j = i$, then $ug^{-1} \in \mathrm{Stab}_G(x^i)$. Observe now that $ug^{-1} \notin U$, otherwise $g = [u^{-1}(ug^{-1})]^{-1} \in U$, against $g \in G \setminus U$ and therefore $\mathrm{Stab}_G(x^i) \neq \mathrm{Stab}_U(x^i)$ which, by i), says that $(x^i)^G = (x^i)^U$. □

For every $g \in G$, the subset of elements of X fixed by g is denoted by

$$\mathrm{Fix}_X(g) = \{x \in X : x^g = x\}.$$

Clearly , given $g \in G$ and $x \in X$, we have that $g \in \mathrm{Stab}_G(x)$ if and only if $x \in \mathrm{Fix}_X(g)$. The number $R(G)$ of G-orbits may be computed by the famous Frobenius Formula making the average on G of the order of the sets $\mathrm{Fix}_X(g)$:

$$R(G) = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}_X(g)|. \tag{9}$$

2.5 Relations

Let X be a set. A relation $\mathcal{R}$ on X is a subset of $X \times X$. We say that a relation $\mathcal{R}_2$ on X extends a relation $\mathcal{R}_1$ on X if $\mathcal{R}_1 \subseteq \mathcal{R}_2$. If $U \subseteq X$ and $\mathcal{R}$ is a relation on X, the *restriction* of $\mathcal{R}$ to U is the relation on U given by $\mathcal{R} \cap (U \times U)$. A relation $\mathcal{R}$ on X is *reflexive* if, for every $x \in X$, $(x,x) \in \mathcal{R}$; *irreflexive* if for every $x \in X$, $(x,x) \notin \mathcal{R}$; *transitive* if, for every $x, y, z \in X$, if $(x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}$, then $(x,z) \in \mathcal{R}$; *complete* if, for every $x, y \in X$, we have $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$; *antisymmetric* if, for every $x, y \in X$, if $(x,y) \in \mathcal{R}$ and $(y,x) \in \mathcal{R}$, then $x = y$; *asymmetric* if, for every $x, y \in X$, if $(x,y) \in \mathcal{R}$ then $(y,x) \notin \mathcal{R}$. Note that if $\mathcal{R}$ is asymmetric, then it is also irreflexive. Given $x, y \in X$, with $x \neq y$, a *chain* for $\mathcal{R}$ (or a $\mathcal{R}$-chain) from x to y is an ordered sequence $x_1, \ldots, x_l$, with $l \in \mathbb{N}$, $l \geq 2$, of distinct elements of X such that $x_1 = x, x_l = y$, and for every $j \in \{1, \ldots, l-1\}$, $(x_j, x_{j+1}) \in \mathcal{R}$; $l-1$ is called the length of the chain, x the starting point and y the end point. A *cycle* in $\mathcal{R}$ of length l is an ordered sequence $x_1, \ldots, x_l$ with $l \in \mathbb{N}$, $l \geq 2$ of distinct elements of X such that, for every $j \in \{1, \ldots, l-1\}$, $(x_j, x_{j+1}) \in \mathcal{R}$ and $(x_l, x_1) \in \mathcal{R}$. A relation $\mathcal{R}$ is called *acyclic* if contains no cycle.

Given a relation $\mathcal{R}$ on X and $x, y \in X$, sometimes we write $x \geq_{\mathcal{R}} y$ instead of $(x,y) \in \mathcal{R}$, and we write $x >_{\mathcal{R}} y$ instead of $(x,y) \in \mathcal{R}$ and $(y,x) \notin \mathcal{R}$. Note that $x >_{\mathcal{R}} y$ implies $x \neq y$ and that if $\mathcal{R}$ is asymmetric, then $x >_{\mathcal{R}} y$ is equivalent to $x \geq_{\mathcal{R}} y$.

A relation on X is called *linear order* on X if it is complete, transitive and antisymmetric. It is well known that a relation can be extended to a linear order if and only if it is acyclic. Clearly the restriction to $U \subseteq X$ of a linear order on X is a linear order on U. The set of linear orders on X is denoted by $\mathcal{L}(X)$

2.6 Linear orders, vectors and permutations

Fix $n \in \mathbb{N}$ and let $N = \{1, \ldots, n\}$. For every $\mathcal{R} \in \mathcal{L}(N)$ and $\psi \in S_n$, denote by $\psi\mathcal{R}$ the relation over N such that, for every $x, y \in N$, $(x,y) \in \psi\mathcal{R}$ if and only if $(\psi^{-1}(x), \psi^{-1}(y)) \in \mathcal{R}$. Of course, $\psi\mathcal{R} \in \mathcal{L}(N)$. Moreover, by definition,

$$x >_{\mathcal{R}} y \quad \Leftrightarrow \quad \psi(x) >_{\psi\mathcal{R}} \psi(y) \tag{10}$$

Let $\rho_0 \in S_n$ be defined, for every $x \in N$, as $\rho_0(x) = n - x + 1$. For every $\mathcal{R} \in \mathcal{L}(N)$ denote by $\mathcal{R}\rho_0$ the relation over N such that, for every $x, y \in N$, $(x,y) \in \mathcal{R}\rho_0$ if and only if $(y,x) \in \mathcal{R}$. Note that $\mathcal{R}\rho_0 \in \mathcal{L}(N)$. Of course,

$$x >_{\mathcal{R}} y \quad \Leftrightarrow \quad y >_{\mathcal{R}\rho_0} x, \tag{11}$$

which means that the preference $\mathcal{R}\rho_0$ is the *reversal* of $\mathcal{R}$. Define also $\mathcal{R}\,id = \mathcal{R}$.

Consider now the set of vectors with n distinct components in N given by

$$\mathcal{A}(N) = \left\{a = (a_j)_{j=1}^n \in N^n : a_{j_1} = a_{j_2} \Rightarrow j_1 = j_2\right\}$$

and think the vector $a = (a_j)_{j=1}^n \in \mathcal{A}(N)$ as the column vector

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [a_1, \ldots, a_n]^T.$$

It is easy to check that the functions

- $f_1 : \mathcal{A}(N) \to \mathcal{L}(N)$ mapping $a = (a_j)_{j=1}^n \in \mathcal{A}(N)$ into $\mathcal{R} \in \mathcal{L}(N)$ such that, for every $a_{j_1}, a_{j_2} \in N$, $(a_{j_1}, a_{j_2}) \in \mathcal{R}$ if and only if $j_1 \leq j_2$,
- $f_2 : S_n \to \mathcal{L}(N)$ mapping $\sigma \in S_n$ into $\mathcal{R} \in \mathcal{L}(N)$ such that, for every $x, y \in N$, $(x,y) \in \mathcal{R}$ if and only if $\sigma^{-1}(x) \leq \sigma^{-1}(y)$.

are bijective. By them we are allowed to identify linear orders with column vectors or permutations when needed.

Consider now $\mathcal{R} \in \mathcal{L}(N)$. Identifying $\mathcal{R}$ with $[a_1, \ldots, a_n]^T \in \mathcal{A}(N)$, we have that $\psi\mathcal{R}$ corresponds to $[\psi(a_1), \ldots, \psi(a_n)]^T$ and $\mathcal{R}\rho_0$ corresponds to $[a_{\rho_0(1)}, \ldots, a_{\rho_0(n)}]^T = [a_n, \ldots, a_1]^T$. Identifying instead $\mathcal{R}$ with $\sigma \in S_n$, we have that $\psi\mathcal{R}$ corresponds to the product $\psi\sigma \in S_n$ and $\mathcal{R}\rho_0$ corresponds to product $\sigma\psi_0 \in S_n$. As a consequence, for every $\mathcal{R} \in \mathcal{L}(N)$, $\psi_1, \psi_2 \in S_n$ and $\psi_3, \psi_4 \in \{id, \rho_0\}$ we have that

- $\psi_1\mathcal{R} = \mathcal{R}$ if and only if $\psi_1 = id$,
- $\mathcal{R}\psi_3 = \mathcal{R}$ if and only if $\psi_3 = id$,
- $\psi_1(\psi_2\mathcal{R}) = (\psi_1\psi_2)\mathcal{R}$, $\psi_1(\mathcal{R}\psi_3) = (\psi_1\mathcal{R})\psi_3$ and $(\mathcal{R}\psi_3)\psi_4 = \mathcal{R}(\psi_3\psi_4)$.

2.7 Formal model and main results

From now on, let $h, n \in \mathbb{N}$ with $h, n \geq 2$ be fixed. Let $H = \{1, \ldots, h\}$ be the set of individuals and $N = \{1, \ldots, n\}$ be the set of alternatives. A *preference* over N is an element of $\mathcal{L}(N)$. Given $p_0 \in \mathcal{L}(N)$ and $x, y \in N$, we say that x is *at least as good as* y according to p_0, if $x \geq_{p_0} y$ and x is *preferred* to y according to p_0 if $x >_{p_0} y$.[9] A *preference profile* is an element of $\mathcal{L}(N)^h$. The set $\mathcal{L}(N)^h$ is denoted by $\mathcal{P}$. If $p \in \mathcal{P}$ and $i \in H$, the i-th component of p is denoted by p_i and represents the preference of individual i. Any $p \in \mathcal{P}$ can be identified with the matrix whose i-th column is the column vector representing the i-th component of p. Note that $|\mathcal{P}| = n!^h$. A profile $p \in \mathcal{P}$ is called *constant* if there exists $q_0 \in \mathcal{L}(N)$ such that $p_i = q_0$ for all $i \in H$. In other words, a profile is constant if all the individuals express the same preference.

A *rule* or *social welfare function* is a function from $\mathcal{P}$ to $\mathcal{L}(N)$. The set of all rules is denoted by $\mathcal{F}$.

Let $\Omega = \langle \rho_0 \rangle$. It is immediate to observe that $\Omega = \{id, \rho_0\}$, so that $|\rho_0| = 2$ and Ω is abelian. Consider then the group $G = S_h \times S_n \times \Omega$ and its subgroup $U = S_h \times S_n \times \{id\}$.

For every $(\varphi, \psi, \rho) \in G$ and $p \in \mathcal{P}$, define $p^{(\varphi,\psi,\rho)} \in \mathcal{P}$ as the preference profile such that, for every $i \in H$,

$$\left(p^{(\varphi,\psi,\rho)}\right)_i = \psi p_{\varphi^{-1}(i)}\rho. \tag{12}$$

The profile $p^{(\varphi,\psi,\rho)}$ is thus the profile obtained by p as if alternatives and individuals were renamed according to the following rules: for every $i \in H$, individual i is renamed $\varphi(i)$; for every $x \in N$, alternative x is renamed $\psi(x)$; if $\rho = id$ nothing else happens, while if $\rho = \rho_0$ the final ranking is reversed top-down. For instance, if $n = 3$, $h = 5$ and

$$p = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 3 & 3 & 1 & 1 \end{bmatrix}, \quad \varphi = (134)(25), \quad \psi = (12), \quad \rho = \rho_0 = (13) \tag{13}$$

we have

$$p^{(\varphi,id,id)} = \begin{bmatrix} 3 & 2 & 3 & 2 & 1 \\ 2 & 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 3 \end{bmatrix}, \quad p^{(id,\psi,id)} = \begin{bmatrix} 3 & 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 2 & 2 \end{bmatrix},$$

$$p^{(id,id,\rho_0)} = \begin{bmatrix} 1 & 3 & 3 & 1 & 1 \\ 2 & 2 & 1 & 2 & 3 \\ 3 & 1 & 2 & 3 & 2 \end{bmatrix}, \quad p^{(\varphi,\psi,\rho_0)} = \begin{bmatrix} 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 2 \end{bmatrix}$$

[9] As p_0 is a linear order, we have that $x >_{p_0} y$ if and only if $x \neq y$ and $x \geq_{p_0} y$.

Since we have given no meaning to $(p_i)^{(\varphi,\psi,\rho)}$ for a single preference $p_i \in S_n$, we will write the i-th component of the profile $p^{(\varphi,\psi,\rho)}$ simply as $p_i^{(\varphi,\psi,\rho)}$, instead of $\left(p^{(\varphi,\psi,\rho)}\right)_i$, because no misleading is possible.

A rule F is said *anonymous and neutral* if, for every $p \in \mathcal{P}$ and $(\varphi, \psi, id) \in U$,

$$F(p^{(\varphi,\psi,id)}) = \psi F(p) id = \psi F(p) \tag{14}$$

The set of anonymous and neutral rules is denoted by $\mathcal{F}^{\text{an}}$. This kind of rules have been considered in Bubboloni and Gori (2013).

A rule F is said *anonymous, neutral and reversal symmetric* if, for every $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in G$,

$$F(p^{(\varphi,\psi,\rho)}) = \psi F(p)\rho, \tag{15}$$

The set of anonymous, neutral and reversal symmetric rules is denoted by $\mathcal{F}^{\text{anr}}$. We prove the following theorem.

Theorem 2. *If* $\gcd(h, n!) = 1$, *then*

$$|\mathcal{F}^{\text{anr}}| \geq \left(2^{\lfloor \frac{n}{2} \rfloor} \left\lfloor \frac{n}{2} \right\rfloor !\right)^{\left\lceil \frac{(h+n!-1)!}{2(n!-1)!n!h!} \right\rceil}$$

If $\gcd(h, n!) \neq 1$, *then* $\mathcal{F}^{\text{anr}} = \varnothing$.

Given $\nu \in \mathbb{N} \cap (h/2, h]$, let us define, for every $p \in \mathcal{P}$, the set

$$C_\nu(p) = \left\{ q_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : x >_{p_i} y\}| \geq \nu \;\; \Rightarrow \;\; x >_{q_0} y \right\},$$

that is, the set of preferences having x preferred to y whenever, according to the preference profile p, at least ν individuals prefer x to y. It is known that[10] the two following conditions are equivalent:

- for every $p \in \mathcal{P}$, $C_\nu(p) \neq \varnothing$,
- $\nu > \frac{n-1}{n}h$,

A rule F is said a *ν-majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_\nu(p)$. The set of ν-majority rules is denoted by $\mathcal{F}_\nu$ and, by the previous result, $\mathcal{F}_\nu \neq \varnothing$ if and only if $\nu > \frac{n-1}{n}h$. If $\nu, \nu' \in \mathbb{N} \cap (h/2, h]$ and $\nu \leq \nu'$, then we have $C_\nu(p) \subseteq C_{\nu'}(p)$, for all $p \in \mathcal{P}$, and thus $\mathcal{F}_\nu \subseteq \mathcal{F}_{\nu'}$.

For every $p \in \mathcal{P}$, define also

$$\nu(p) = \min\{\nu \in \mathbb{N} \cap (h/2, h] : C_\nu(p) \neq \varnothing\}.$$

Of course, $\nu(p)$ is well defined as, for every $p \in \mathcal{P}$, $C_h(p) \neq \varnothing$. Moreover $\nu(p) \leq \lceil \frac{n-1}{n}h \rceil$.

A rule F is said a *minimal majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_{\nu(p)}(p)$. The set of minimal majority rules, denoted by $\mathcal{F}_{\min}$, is nonempty and $\mathcal{F}_{\min} \subseteq \mathcal{F}_\nu$ for all $\nu > \frac{n-1}{n}h$.

Define the sets $\mathcal{F}^{\text{anr}}_{\min} = \mathcal{F}^{\text{anr}} \cap \mathcal{F}_{\min}$ and $\mathcal{F}^{\text{anr}}_\nu = \mathcal{F}^{\text{anr}} \cap \mathcal{F}_\nu$. Note that, if $\nu > \frac{n-1}{n}h$, then $\mathcal{F}^{\text{anr}}_{\min} \subseteq \mathcal{F}^{\text{anr}}_\nu$. The following theorems hold.

Theorem 3. $\mathcal{F}^{\text{anr}}_{\min} \neq \varnothing$ *if and only if* $\gcd(h, n!) = 1$.

Theorem 4. $\mathcal{F}^{\text{anr}}_\nu \neq \varnothing$ *if and only if* $\gcd(h, n!) = 1$ *and* $\nu > \frac{n-1}{n}h$.

[10] See Can and Storken (2012) and Theorem 10 in Bubboloni and Gori (2013)

3 Proofs of the main existence results

3.1 Actions and stabilizers

We start with the description of some elementary properties of ρ_0 and Ω. Recalling that $\rho_0 \in S_n$ is defined, for every $x \in N$, by $\rho_0(x) = n - x + 1$ and that $\Omega = \{id, \rho_0\}$, its proof is immediate. For ii) we invoke a well known result in group theory allowing to determine the centralizer of a permutation, when the type of the permutation is known.

Lemma 5. *Let $\lambda \vdash n$ be the type of ρ_0. Then:*

i) if n is even, $\lambda_i = 2$ for every part of λ. If n is odd, $\lambda_i = 2$ for every part of λ except $\lambda_{\frac{n+1}{2}} = 1$ and the only fixed point of ρ_0 is $x_0 = \frac{n+1}{2}$;

ii) $\Omega \leq C_{S_n}(\rho_0)$ and $C_{S_n}(\rho_0)$ is isomorph to $S_2 \wr S_{\lfloor \frac{n}{2} \rfloor}$. In particular, $|C_{S_n}(\rho_0)| = 2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !$;

iii) if $n = 2, 3$, then $C_{S_n}(\rho_0) = \Omega$.

The next basic result allows us to exploit many facts from group theory. Recall that G always denotes the group $S_h \times S_n \times \Omega$ and U its normal subgroup $U = S_h \times S_n \times \{id\}$.

Proposition 6. *The function $f : G \to \mathfrak{F}(\mathcal{P})$ defined, for every $(\varphi, \psi, \rho) \in G$, as*

$$f(\varphi, \psi, \rho) : \mathcal{P} \to \mathcal{P}, \quad p \mapsto p^{(\varphi,\psi,\rho)},$$

maps G into $\mathrm{Sym}(\mathcal{P})$ and is an action of G on $\mathcal{P}$.

Proof. First of all, we note that, by definition (12), we have $f(id, id, id) = id$. Then we show that, for every $(\varphi_1, \psi_1, \rho_1), (\varphi_2, \psi_2, \rho_2) \in G$,

$$f((\varphi_1, \psi_1, \rho_1)(\varphi_2, \psi_2, \rho_2)) = f(\varphi_1, \psi_1, \rho_1) f(\varphi_2, \psi_2, \rho_2), \tag{16}$$

that is, for every $p \in \mathcal{P}$ and $(\varphi_1, \psi_1, \rho_1)$, $(\varphi_2, \psi_2, \rho_2) \in G$,

$$p^{(\varphi_1\varphi_2, \psi_1\psi_2, \rho_1\rho_2)} = \left(p^{(\varphi_2,\psi_2,\rho_2)}\right)^{(\varphi_1,\psi_1,\rho_1)}. \tag{17}$$

Indeed, for every $i \in H$, by definition (12) we have

$$p_i^{(\varphi_1\varphi_2, \psi_1\psi_2, \rho_1\rho_2)} = \psi_1\psi_2 p_{(\varphi_1\varphi_2)^{-1}(i)} \rho_1\rho_2$$

and also, recalling that Ω is abelian

$$\left(p^{(\varphi_2,\psi_2,\rho_2)}\right)_i^{(\varphi_1,\psi_1,\rho_1)} = \psi_1 \left(p^{(\varphi_2,\psi_2,\rho_2)}\right)_{\varphi^{-1}(i)} \rho_1 = \psi_1\psi_2 p_{\varphi_2^{-1}(\varphi_1^{-1}(i))} \rho_2\rho_1 = \psi_1\psi_2 p_{(\varphi_1\varphi_2)^{-1}(i)} \rho_1\rho_2$$

As a consequence, for every $(\varphi, \psi, \rho) \in G$, we get

$$f(\varphi, \psi, \rho) f(\varphi^{-1}, \psi^{-1}, \rho) = f(\varphi^{-1}, \psi^{-1}, \rho) f(\varphi, \psi, \rho) = f(id, id, id) = id.$$

Thus, $f(\varphi, \psi, \rho)$ is a function of $\mathcal{P}$ into itself with inverse $f(\varphi^{-1}, \psi^{-1}, \rho)$, and therefore $f(\varphi, \psi, \rho) \in \mathrm{Sym}(\mathcal{P})$. Finally note that the fact that f is a homomorphism from the group G into the group $\mathrm{Sym}(\mathcal{P})$, is now exactly the content of equality (16). □

The action of G on $\mathcal{P}$ considered in Proposition 6 behave very naturally with respect to various parameters introduced in Section 2.7.

Proposition 7. *Let $\nu \in \mathbb{N} \cap (h/2, h]$, $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in G$. Then:*

i) $C_\nu(p^{(\varphi,\psi,\rho)}) = \psi C_\nu(p)\rho$;

ii) $\nu(p^{(\varphi,\psi,\rho)}) = \nu(p)$;

iii) $C_{\nu(p^{(\varphi,\psi,\rho)})}(p^{(\varphi,\psi,\rho)}) = \psi C_{\nu(p)}(p)\rho$.

Proof. i) By Lemma 11 in Bubboloni and Gori (2013) we know that $C_\nu(p^{(\varphi,\psi,id)}) = \psi C_\nu(p)$ for all $p \in \mathcal{P}$ and all $(\varphi, \psi, id) \in U$. Since $C_\nu(p^{(\varphi,\psi,\rho_0)}) = C_\nu(\left[p^{(\varphi,\psi,id)}\right]^{(id,id,\rho_0)})$, we just need to show that, for all $p \in \mathcal{P}$

$$C_\nu(p^{(id,id,\rho_0)}) = C_\nu(p)\rho_0.$$

But, due to $p_i^{(id,id,\rho_0)} = p_i\rho_0$ and recalling that $|\rho_0| = 2$, we have immediately

$$\begin{aligned} C_\nu(p^{(id,id,\rho_0)}) &= \{q_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : x >_{p_i\rho_0} y\}| \geq \nu \Rightarrow x >_{q_0} y\} \\ &= \{q_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : y >_{p_i} x\}| \geq \nu \Rightarrow y >_{q_0\rho_0} x\} \\ &= \{q_1\rho_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : y >_{p_i} x\}| \geq \nu \Rightarrow y >_{q_1} x\} \\ &= C_\nu(p)\rho_0. \end{aligned}$$

ii)-iii) By i) we get that $|C_\nu(p^{(\varphi,\psi,\rho)})| = |C_\nu(p)|$ and thus $C_\nu(p^{(\varphi,\psi,\rho)}) \neq \varnothing$ if and only if $C_\nu(p) \neq \varnothing$, that is $\nu(p^{(\varphi,\psi,\rho)}) = \nu(p)$. It follows also $C_{\nu(p^{(\varphi,\psi,\rho)})}(p^{(\varphi,\psi,\rho)}) = C_{\nu(p)}(p^{(\varphi,\psi,\rho)}) = \psi C_{\nu(p)}(p)\rho$. □

The restriction to U of the action of G considered in Proposition 6 coincides, up to the natural identification of U with $S_h \times S_n$, with the action considered in Bubboloni and Gori (2013) to describe anonymity and neutrality. This enables us to obtain a result about the action of U as an easy consequence of Proposition 2 in Bubboloni and Gori (2013).

Lemma 8. *Let* $\gcd(h, n!) = 1$, $p \in \mathcal{P}$ *and* $(\varphi, \psi, id) \in U$ *Then:*

i) $\mathrm{Fix}_\mathcal{P}(\varphi, \psi, id) \neq \varnothing$ *if and only if* $\psi = id$;

ii) $\mathrm{Stab}_U(p) \leq S_h \times \{id\} \times \{id\}$.

Proof. i) Let $(\varphi, \psi, id) \in U$ and consider $\mathrm{Fix}_\mathcal{P}(\varphi, \psi, id)$. This set coincides with the fix set $\mathrm{Fix}_\mathcal{P}(\varphi, \psi)$ related to the action of $S_h \times S_n$ on $\mathcal{P}$ considered in Bubboloni and Gori (2013), so that we can apply Proposition 2 in Bubboloni and Gori (2013), to deduce that $\mathrm{Fix}_\mathcal{P}(\varphi, \psi, id)$ is non-empty if and only if $|\psi| \mid \gcd(\lambda)$, where $\lambda \vdash h$ is the type of φ.

We examine first $\mathrm{Fix}_\mathcal{P}(\varphi, \psi, id)$ with $\psi = id$. Since $1 = |\psi|$ obviously divides $\gcd(\lambda)$, then $\mathrm{Fix}_\mathcal{P}(\varphi, id, id) \neq \varnothing$. Conversely if $\mathrm{Fix}_\mathcal{P}(\varphi, \psi, id) \neq \varnothing$, then $|\psi| \mid \gcd(\lambda)$ and therefore $|\psi| \mid h$. But $\psi \in S_n$ gives also $|\psi| \mid n!$ and, by the assumption $\gcd(h, n!) = 1$, we get $|\psi| = 1$, that is $\psi = id$.

ii) Let $g = (\varphi, \psi, id) \in \mathrm{Stab}_U(p)$. Then $p \in \mathrm{Fix}_\mathcal{P}(\varphi, \psi, id) \neq \varnothing$ and, by i), we get $\psi = id$ so that $g = (\varphi, id, id) \in S_h \times \{id\} \times \{id\}$. □

We now describe, for $p \in \mathcal{P}$, the subgroup $\mathrm{Stab}_G(p)$ and the order of the orbit p^G.

Proposition 9. *Let* $\gcd(h, n!) = 1$ *and* $p \in \mathcal{P}$. *The following holds:*

a) If $\mathrm{Stab}_G(p) = \mathrm{Stab}_U(p)$, *then* $|p^G| = 2|p^U|$.

b) If $\mathrm{Stab}_G(p) > \mathrm{Stab}_U(p)$ *and* $(\varphi, \psi, \rho_0) \in \mathrm{Stab}_G(p)$, *then:*

i) there exists $u \in S_n$ *such that* $\psi = u\rho_0 u^{-1}$;

ii) ψ *has order* 2 *and the same type of* ρ_0. *In particular* ψ *has no fixed point if* n *is even and just one if* n *is odd;*

iii) $\mathrm{Stab}_G(p) = \mathrm{Stab}_U(p)\langle(\varphi,\psi,\rho_0)\rangle \leq S_h \times \langle(\psi,\rho_0)\rangle$*, where* $\langle(\psi,\rho_0)\rangle = \{(\psi,\rho_0),(id,id)\}$*. Moreover* $|\mathrm{Stab}_G(p)| = 2\,|\mathrm{Stab}_U(p)|$ *and* $|p^G| = |p^U|$*;*

iv) if $(\varphi_1,\psi_1,\rho_0) \in \mathrm{Stab}_G(p)$*, then* $\psi_1 = \psi$*.*

Proof. a) Apply Lemma 1 to $X = \mathcal{P}$.

b) i)-ii). Let $p \in \mathcal{P}$ and $g = (\varphi,\psi,\rho_0) \in \mathrm{Stab}_G(p)$. Since $\mathrm{Stab}_G(p)$ is a subgroup of G, we have also

$$g^2 = (\varphi^2,\psi^2,id) \in \mathrm{Stab}_G(p) \cap U = \mathrm{Stab}_U(p)$$

and thus, by Lemma 8ii), $\psi^2 = id$, that is $\psi = id$ or $|\psi| = 2$.

Let $\varphi = \gamma_1 \cdots \gamma_r$, be a decomposition of φ into disjoint cycles γ_i of length λ_i and let $\lambda \vdash h$ be the corresponding partition of h. Since $n \geq 2$ and $\gcd(h,n!) = 1$, we have that h is odd an thus the set $O(\varphi) = \{i \in H : \lambda_i \text{ is odd}\}$ is non-empty. Take $i \in O(\varphi)$ and note that φ^{λ_i} has at least $\lambda_i \geq 1$ fixed points, given by the $j \in H$ involved by the cycle γ_i. Moreover we have

$$g^{\lambda_i} = (\varphi^{\lambda_i},\psi,\rho_0) \in \mathrm{Stab}_G(p), \tag{18}$$

because the odd power of an element of order 2 coincides with the element itself.

But (18) means that for all $j \in H$, $p_j^{(\varphi^{\lambda_i},\psi,\rho_0)} = p_j$, that is

$$\psi p_j \rho_0 = p_{\varphi^{\lambda_i}(j)}. \tag{19}$$

In particular, for all $j \in H$ involved in the cycle γ_i, we get $\psi p_j \rho_0 = p_j$ and so $\psi = p_j(p_j\rho_0)^{-1} = p_j\rho_0^{-1}p_j^{-1} = p_j\rho_0 p_j^{-1}$. Thus i) follows taking $u = p_j$, for a certain $j \in H$ involved in the cycle γ_i. Note that we have proved that ψ is a conjugate of ρ_0 [11]. By a well known group theory result, conjugate permutations have the same type and, in particular, have the same order. It follows that ψ has the same type of ρ_0 and $|\psi| = 2$, so that ii) follows.

iii) This is an immediate consequence of Lemmas 1 and 8, recalling that $[G : U] = 2$.

iv) From $(\varphi,\psi,\rho_0),(\varphi_1,\psi_1,\rho_0) \in \mathrm{Stab}_G(p)$, we get $(\varphi,\psi,\rho_0)(\varphi_1,\psi_1,\rho_0)^{-1} \in \mathrm{Stab}_G(p)$, that is $(\varphi\varphi_1^{-1},\psi\psi_1^{-1},id) \in \mathrm{Stab}_U(p)$ which by Lemma 8 gives $\psi_1 = \psi$. □

3.2 Proof of Theorem 2

For shortness, from now on, when the acting group is G we will speak of orbits instead of G-orbits and more generally, in all notation, we will omit any explicit reference to the group G. In particular $\mathfrak{S}$ will stand for $\mathfrak{S}(G)$ and R for $R(G)$.

Define, for every $p \in \mathcal{P}$,

$$S^0(p) = \{q_0 \in \mathcal{L}(N) : \forall(\varphi,\psi,\rho) \in \mathrm{Stab}_G(p), \psi q_0 \rho = q_0\}.$$

This set is the natural codomain for the anonymous, neutral and reversal rules.

Lemma 10. *Let* $F \in \mathcal{F}^{\mathrm{anr}}$*. Then, for every* $p \in \mathcal{P}$*,* $F(p) \in S^0(p)$*.*

Proof. Let $p \in \mathcal{P}$ and $(\varphi,\psi,\rho) \in \mathrm{Stab}_G(p)$. Then $p = p^{(\varphi,\psi,\rho)}$ and so $F(p) = F(p^{(\varphi,\psi,\rho)}) = \psi F(p)\rho$, which says $F(p) \in S^0(p)$. □

Proposition 11. *Let* $(p^j)_{j=1}^R \in \mathfrak{S}$ *and* $(q_j)_{j=1}^R \in \times_{j=1}^R S^0(p^j)$*. Then, there exists a unique* $F \in \mathcal{F}^{\mathrm{anr}}$ *such that, for every* $j \in \{1,\ldots,R\}$*,* $F(p^j) = q_j$*. We denote this rule by* $\Phi\left[(p^j)_{j=1}^R,(q_j)_{j=1}^R\right]$*.*

[11] Due to our right to left notation, the conjugate of ρ_0 by u is written $u\rho_0u^{-1}$. Recall that, within the left to right notation used in permutation group theory, the conjugate of ρ_0 by u is written $u^{-1}\rho_0u$. Their meaning is the same, because in both notation u^{-1} applies first.

Proof. We know that $\{p^{j\,G} : j \in \{1,\dots,R\}\}$ is a partition of $\mathcal{P}$. Then, given $p \in \mathcal{P}$, there exist a unique $j \in \{1,\dots,R\}$ such that $p \in p^{j\,G}$. We claim first that, fixed $j \in \{1,\dots,R\}$, if there exist $(\varphi_1,\psi_1,\rho_1),(\varphi_2,\psi_2,\rho_2) \in G$ with $p^{j\,(\varphi_1,\psi_1,\rho_1)} = p^{j\,(\varphi_2,\psi_2,\rho_2)}$, then $\psi_1 q_j \rho_1 = \psi_2 q_j \rho_2$. In fact, by the equality (17), $p^{j\,(\varphi_1,\psi_1,\rho_1)} = p^{j\,(\varphi_2,\psi_2,\rho_2)}$ implies that $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1, \rho_2^{-1}\rho_1) \in \mathrm{Stab}_G(p^j)$. Since $q_j \in S^0(p^j)$ and Ω is abelian, we have then $q_j = \psi_2^{-1}\psi_1 q_j \rho_2^{-1}\rho_1 = \psi_2^{-1}\psi_1 q_j \rho_1\rho_2^{-1}$ and so $\psi_1 q_j \rho_1 = \psi_2 q_j \rho_2$, as desired.

Consider now the rule F defined, for every $p \in \mathcal{P}$, as $F(p) = \psi q_j \rho$, where $j \in \{1,\dots,R\}$ and $(\varphi,\psi,\rho) \in G$ are such that $p = p^{j\,(\varphi,\psi,\rho)}$. Note that, because of our previous claim, the definition of F is unambiguous. Let us prove now that $F \in \mathcal{F}^{\mathrm{anr}}$, seeing that for each $p \in \mathcal{P}$ and $(\varphi,\psi,\rho) \in G$ we have $F(p^{(\varphi,\psi,\rho)}) = \psi F(p)\rho$. Indeed, let $p = p^{j\,(\varphi_*,\psi_*,\rho_*)}$ for some $j \in \{1,\dots,R\}$ and $(\varphi_*,\psi_*,\rho_*) \in G$. By definition of F and by the equality (17), we have

$$\begin{aligned} F(p^{(\varphi,\psi,\rho)}) &= F\left(\left(p^{j\,(\varphi_*,\psi_*,\rho_*)}\right)^{(\varphi,\psi,\rho)}\right) = F(p^{j\,(\varphi\varphi_*,\psi\psi_*,\rho\rho_*)}) = (\psi\psi_*)q_j(\rho\rho_*) \\ &= (\psi\psi_*)q_j(\rho_*\rho) = \psi(\psi_* q_j \rho_*)\rho = \psi F(p^{j\,(\varphi_*,\psi_*,\rho_*)})\rho \\ &= \psi F(p)\rho. \end{aligned}$$

In order to prove the uniqueness of F, it suffices to note that if $F' \in \mathcal{F}^{\mathrm{anr}}$ is such that, for every $j \in \{1,\dots,R\}$, $F'(p^j) = q_j$, then F' must satisfy also $F'(p^{j\,(\varphi,\psi,\rho)}) = \psi q_j \rho = F(p^{j\,(\varphi,\psi,\rho)})$ for all $(\varphi,\psi,\rho) \in G$ and thus $F'(p) = F(p)$ for all $p \in \mathcal{P}$. □

Proposition 12. *Let $(p^j)_{j=1}^R \in \mathfrak{S}$. Then the function*

$$f : \times_{j=1}^R S^0(p^j) \to \mathcal{F}^{\mathrm{anr}}, \quad f\left((q_j)_{j=1}^R\right) = \Phi\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right] \tag{20}$$

is bijective. In particular, $|\mathcal{F}^{\mathrm{anr}}| = \prod_{j=1}^R |S^0(p^j)|$.

Proof. The function f is injective because if for $(q_j)_{j=1}^R, (q'_j)_{j=1}^R \in \times_{j=1}^R S^0(p^j)$ we have $f\left((q_j)_{j=1}^R\right) = f\left((q'_j)_{j=1}^R\right)$, then, for every $j \in \{1,\dots,R\}$, $q_j = f\left((q_j)_{j=1}^R\right)(p^j) = f\left((q'_j)_{j=1}^R\right)(p^j) = q'_j$. Moreover, f is surjective because Lemma 10 and Proposition 11 imply that, for every $F \in \mathcal{F}^{\mathrm{anr}}$, $F = f\left((F(p^j))_{j=1}^R\right)$. □

Proposition 13. *Let $\gcd(h, n!) = 1$ and $p \in \mathcal{P}$.*

a) If $\mathrm{Stab}_G(p) = \mathrm{Stab}_U(p)$, then $S^0(p) = \mathcal{L}(N) = S_n$.

b) If $\mathrm{Stab}_G(p) > \mathrm{Stab}_U(p)$, then $S^0(p) \neq \varnothing$.
Moreover, if $(\varphi,\psi,\rho_0) \in \mathrm{Stab}_G(p)$ and $u \in S_n$ realizes $\psi = u\rho_0 u^{-1}$, then

$$S^0(p) = \{q_0 \in \mathcal{L}(N) : \psi q_0 \rho_0 = q_0\} = uC_{S_n}(\rho_0).$$

In particular, $|S^0(p)| \geq 2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !$.

Proof. We start observing that, by definition,

$$S^0(p) = \bigcap_{(\varphi,\psi,\rho)\in \mathrm{Stab}_G(p)} \{q_0 \in \mathcal{L}(N) : \psi q_0 \rho = q_0\}. \tag{21}$$

If $(\varphi,\psi,\rho) \in \mathrm{Stab}_U(p)$, then we have $\rho = id$ and, by Lemma 8, also $\psi = id$ so that $\{q_0 \in \mathcal{L}(N) : \psi q_0 \rho = q_0\} = \mathcal{L}(N)$. In other words the elements in $\mathrm{Stab}_U(p)$ are ineffective in (21). In particular if $\mathrm{Stab}_G(p) = \mathrm{Stab}_U(p)$, then $S^0(p) = \mathcal{L}(N)$ and a) is proved.

Moreover, if there exists some $(\varphi_*,\psi_*,\rho_0) \in \mathrm{Stab}_G(p)$, then by Lemma 9 iv) all the elements in $\mathrm{Stab}_G(p) \setminus \mathrm{Stab}_U(p)$ share the same second component ψ_* and thus

$$S^0(p) = \bigcap_{(\varphi,\psi,\rho)\in [\mathrm{Stab}_G(p)\setminus \mathrm{Stab}_U(p)]} \{q_0 \in \mathcal{L}(N) : \psi q_0 \rho = q_0\} = \{q_0 \in \mathcal{L}(N) : \psi_* q_0 \rho_0 = q_0\}.$$

By Lemma 9 b), there exists $u \in S_n$ with $\psi_* = u\rho_0 u^{-1}$. We want to show that

$$\{q_0 \in \mathcal{L}(N) : \psi_* q_0 \rho_0 = q_0\} = uC_{S_n}(\rho_0).$$

Note that $q_0 \in \{q_0 \in \mathcal{L}(N) : \psi_* q_0 \rho_0 = q_0\} \quad \Leftrightarrow \quad \psi_* = q_0\rho_0 q_0^{-1} \quad \Leftrightarrow \quad \rho_0 u^{-1} q_0 = u^{-1} q_0 \rho_0 \quad \Leftrightarrow$ $u^{-1}q_0 \in C_{S_n}(\rho_0) \quad \Leftrightarrow \quad q_0 \in uC_{S_n}(\rho_0)$. Thus $S^0(p) = uC_{S_n}(\rho_0)$ and b) is proved. □

Proof of Theorem 2. Assume $\gcd(h, n!) = 1$ and fix $(p^j)_{j=1}^R \in \mathfrak{S}$. By Proposition 13 we have that, for every $j \in \{1, \ldots, R\}$, $|S^0(p^j)| \geq 2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor!$. Note also that by Lemma 1 ii) and Proposition 23 in Bubboloni and Gori (2013) we have that $R \geq \left\lceil \frac{R(U)}{2} \right\rceil = \left\lceil \frac{(h+n!-1)!}{2(n!-1)!n!h!} \right\rceil$. As a consequence, by Lemma 5ii), we have that

$$|\mathcal{F}^{\text{anr}}| = \prod_{j=1}^{R} |S^0(p^j)| \geq \left(2^{\lfloor \frac{n}{2} \rfloor} \left\lfloor \frac{n}{2} \right\rfloor!\right)^{\left\lceil \frac{(h+n!-1)!}{2(n!-1)!n!h!} \right\rceil} \geq 2.$$

In particular $\mathcal{F}^{\text{anr}} \neq \varnothing$.

Assume now $\gcd(h, n!) \neq 1$. Then by Theorem 6 in Bubboloni and Gori (2013) we have that $\mathcal{F}^{\text{anr}} = \varnothing$. □

3.3 Proof of Theorem 3

Define now, for every $p \in \mathcal{P}$, the subset of $S^0(p)$

$$S^1(p) = S^0(p) \cap C_{\nu(p)}(p).$$

This set is the natural codomain for the anonymous, neutral and reversal minimal majority rules.

Lemma 14. *Let $F \in \mathcal{F}^{\text{anr}}_{\min}$. Then, for every $p \in \mathcal{P}$, $F(p) \in S^1(p)$.*

Proof. Let $p \in \mathcal{P}$. Since $F \in \mathcal{F}^{\text{anr}}$, by Lemma 10, we know that $F(p) \in S^0(p)$. Moreover, as $F \in \mathcal{F}_{\min}$ we also have that $F(p) \in C_{\nu(p)}(p)$. Then $F(p) \in S^1(p)$. □

Proposition 15. *Let $(p^j)_{j=1}^R \in \mathfrak{S}$ and f as defined in (20). Then $f\left(\times_{j=1}^R S^1(p^j)\right) = \mathcal{F}^{\text{anr}}_{\min}$. In particular, $|\mathcal{F}^{\text{anr}}_{\min}| = \prod_{j=1}^R |S^1(p^j)|$.*

Proof. Let us prove first that $f\left(\times_{j=1}^R S^1(p^j)\right) \subseteq \mathcal{F}^{\text{anr}}_{\min}$. Fix $(q_j)_{j=1}^R \in \times_{j=1}^R S^1(p^j)$, define $F = f\left((q_j)_{j=1}^R\right)$ and prove that $F \in \mathcal{F}^{\text{anr}}_{\min}$. By Proposition 12, we have $F \in \mathcal{F}^{\text{anr}}$ and thus to prove $F \in \mathcal{F}_{\min}$, we need only to show that, for every $p \in \mathcal{P}$, $F(p) \in C_{\nu(p)}(p)$. Let $p \in \mathcal{P}$; then for suitable $j \in \{1, \ldots, R\}$ and $(\varphi, \psi, \rho) \in G$, we have $p = p^{j\,(\varphi,\psi,\rho)}$. Thus, being $F \in \mathcal{F}^{\text{anr}}$, we get

$$F(p) = F\left(p^{j\,(\varphi,\psi,\rho)}\right) = \psi F(p^j)\rho = \psi q_j \rho.$$

As $q_j \in C_{\nu(p^j)}(p^j)$, by Proposition 7, we have also

$$F(p) = \psi q_j \rho \in \psi C_{\nu(p^j)}(p^j)\rho = C_{\nu(p^{j\,(\varphi,\psi,\rho)})}(p^{j\,(\varphi,\psi,\rho)}) = C_{\nu(p)}(p),$$

as desired.

The inclusion $\mathcal{F}^{\text{anr}}_{\min} \subseteq f\left(\times_{j=1}^R S^1(p^j)\right)$ follows immediately from Lemma 14 and Proposition 11. □

Proposition 16. *Let $\gcd(h, n!) = 1$. Then, for every $p \in \mathcal{P}$, we have $S^1(p) \neq \varnothing$.*

Proof. See the Appendix. □

Proof of Theorem 3. Assume that $\mathcal{F}^{\text{anr}}_{\min} \neq \varnothing$. Then $\mathcal{F}^{\text{an}} \neq \varnothing$ and, by Theorem 6 in Bubboloni and Gori (2013), we get $\gcd(h, n!) = 1$. Assume now $\gcd(h, n!) = 1$ and fix $(p^j)_{j=1}^R \in \mathfrak{S}$. By Proposition 16 we have that $\times_{j=1}^R S^1(p^j) \neq \varnothing$ and by Proposition 15 we have that $\mathcal{F}^{\text{anr}}_{\min} \neq \varnothing$, as well. □

3.4 Proof of Theorem 4

For every $\nu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$, define the subset of $S^0(p)$

$$S^1_\nu(p) = S^0(p) \cap C_\nu(p).$$

Note that $S^1_{\nu(p)}(p) = S^1$. This set is the natural codomain for the anonymous, neutral and reversal ν-majority rules.

Lemma 17. *Let $\nu \in \mathbb{N} \cap (h/2, h]$ and $F \in \mathcal{F}^{\rm anr}_\nu$. Then, for every $p \in \mathcal{P}$, $F(p) \in S^1_\nu(p)$.*

Proof. Analogous to the proof of Lemma 14. □

Proposition 18. *Let $(p^j)_{j=1}^R \in \mathfrak{S}$ and f defined in (20). Then $f\left(\times_{j=1}^R S^1_\nu(p^j)\right) = \mathcal{F}^{\rm anr}_\nu$. In particular, $|\mathcal{F}^{\rm anr}_\nu| = \prod_{j=1}^R |S^1_\nu(p^j)|$.*

Proof. Analogous to the proof of Proposition 15. □

Proposition 19. *Let $\gcd(h, n!) = 1$ and $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu > \frac{n-1}{n}h$. Then, for every $p \in \mathcal{P}$, $S^1_\nu(p) \neq \varnothing$.*

Proof. Note that, since $\nu > \frac{n-1}{n}h \geq \nu(p)$, we have $S^1(p) \subseteq S^1_\nu(p)$ and apply Proposition 16. □

Proof of Theorem 4. Assume that $\mathcal{F}^{\rm anr}_\nu \neq \varnothing$. Then $\mathcal{F}^{\rm an} \neq \varnothing$ and by Theorem 6 in Bubboloni and Gori (2013) we get $\gcd(h, n!) = 1$. Moreover, we have that $\mathcal{F}_\nu \neq \varnothing$ and by Theorem 10 in Bubboloni and Gori (2013) we have also $\nu > \frac{n-1}{n}h$. Assume now $\gcd(h, n!) = 1$ and $\nu > \frac{n-1}{n}h$ and fix $(p^j)_{j=1}^R \in \mathfrak{S}$. By Proposition 19 we have that $\times_{j=1}^R S^1_\nu(p^j) \neq \varnothing$ and by Proposition 18 we have that $\mathcal{F}^{\rm anr}_\nu \neq \varnothing$. □

4 Counting the rules: the case $n = 3$ and $h = 5$

Propositions 12, 15,18 can be used to count the rules. Of course we need, first of all, to have a system of representatives for the G-orbits and this task is made effective by the knowledge of a system of representatives for the U-orbits, thanks to Lemma 1 iii). Clearly the analysis of the U action, developed in Bubboloni and Gori (2013), especially the notion of block type, gives the tools to find a system of representatives for the U-orbits. In other words, to treat a practical counting of rules we need to take into account both the facts developed in Bubboloni and Gori (2013) and those in the present paper. Below we show how to deal with the case $n = 3$ and $h = 5$. Here $\rho_0 = (13)$, $\Omega = \{(13), id\}$, $U = S_5 \times S_3 \times \{id\}$ and $G = S_5 \times S_3 \times \Omega$. Moreover $\nu \in \{3, 4, 5\}$. In Bubboloni and Gori (2013, sec. 10.2), we have computed $R(U) = 42$, constructed an explicit list of representatives $P = (p^i)_{i=1}^{42} \in \mathfrak{S}(U)$ for the U-orbits, determined the sets $C_\nu(p^i)$ and used this information to get the orders of the set of anonymous and neutral rules as well as those of anonymous and neutral ν-majority rules. Now we extract from P an explicit list $Q = (q^j)_{j=1}^{R(G)} \in \mathfrak{S}(G)$ of representatives for the G-orbits, we compute $R(G) = 26$ and count the number of the various rules introduced in the paper, after having determined the orders of the sets $S^0(q^j)$ and $S^1(q^j)$ for $j = 1, \ldots, 26$. Our results are the following:

$$q^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, \quad q^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, \quad q^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix},$$

$$q^4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, \quad q^5 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, \quad q^6 = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix}$$

$$q^7 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix}, \quad q^8 = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, \quad q^9 = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix},$$

$$q^{10} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, \quad q^{11} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, \quad q^{12} = \begin{bmatrix} 1 & 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix},$$

$$q^{13} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, \quad q^{14} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 1 & 2 \\ 3 & 3 & 3 & 3 & 1 \end{bmatrix}, \quad q^{15} = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix},$$

$$q^{16} = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 1 & 1 & 3 \end{bmatrix}, \quad q^{17} = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 1 \\ 3 & 3 & 1 & 1 & 3 \end{bmatrix}, \quad q^{18} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & 2 \\ 3 & 3 & 1 & 1 & 1 \end{bmatrix},$$

$$q^{19} = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 2 & 2 & 1 \end{bmatrix}, \quad q^{20} = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 & 3 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, \quad q^{21} = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{bmatrix},$$

$$q^{22} = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}, \quad q^{23} = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 3 & 3 \\ 3 & 3 & 3 & 2 & 1 \end{bmatrix}, \quad q^{24} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix},$$

$$q^{25} = \begin{bmatrix} 1 & 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 1 & 1 & 2 \end{bmatrix}, \quad q^{26} = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 3 \\ 3 & 3 & 1 & 2 & 1 \end{bmatrix}$$

	C_3	C_4	C_5	S^0	S^1
q^1	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T\}$
q^2	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^3	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T\}$
q^4	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^5	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^6	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T\}$
q^7	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^8	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^9	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T\}$
q^{10}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{11}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{12}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{13}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T\}$
q^{14}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[2,1,3]^T\}$
q^{15}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{16}	$\{[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[2,1,3]^T\}$
q^{17}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\{[2,1,3]^T,[3,1,2]^T\}$	$\{[2,1,3]^T\}$
q^{18}	$\{[2,3,1]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[2,1,3]^T\}$
q^{19}	$\varnothing$	$\{[1,2,3]^T,[1,3,2]^T,[3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,3,2]^T,[2,3,1]^T\}$	$\{[1,3,2]^T\}$
q^{20}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[3,2,1]^T,[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T\}$
q^{21}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T,[2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[2,1,3]^T\}$
q^{22}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{23}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T,[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{24}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T\}$
q^{25}	$\varnothing$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[1,2,3]^T,[3,2,1]^T\}$	$\{[1,2,3]^T,[3,2,1]^T\}$
q^{26}	$\{[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$	$\{[2,1,3]^T,[3,1,2]^T\}$	$\{[2,1,3]^T\}$

$$|\mathcal{F}^{\mathrm{anr}}| = 2^{26}3^{16}, \quad |\mathcal{F}_5^{\mathrm{anr}}| = 2^{20}3^{14}, \quad |\mathcal{F}_4^{\mathrm{anr}}| = 2^{13}3^{7}, \quad |\mathcal{F}_3^{\mathrm{anr}}| = 0, \quad |\mathcal{F}_{\min}^{\mathrm{anr}}| = 2.$$

To determine the list $Q = (q^j)_{j=1}^{R(G)} \in \mathfrak{S}(G)$ of representatives for the G-orbits, we scroll the list $P = (p^i)_{i=1}^{42}$ starting from the beginning, inquiring if a certain p^i has a stabilizer containing an element of the type $(\varphi, \psi, (13)) \in G$, with ψ an element of order 2 in S_3, that is $\psi \in \{(12), (13), (23)\}$.

If this happens then $[\mathrm{Stab}_G(x) : \mathrm{Stab}_U(x)] = 2$ and so $(p^i)^G = (p^i)^U$: in this case we put $q = p^i$ in the list Q and compute $S^0(q) = \{q_0 \in \mathcal{L}(\{1,2,3\}) : \psi q_0(13) = q_0\}$.

If this does not happen then there exists an unique $j \neq i$, with $1 \leq j \leq 42$ such that $(p^i)^G = (p^i)^U \cup (p^j)^U$: in this case we set $i_0 = \min\{i, j\}$, put $q = p_{i_0}$ in the list Q, eliminate p^j from the list P and deduce that $S^0(q) = \mathcal{L}(\{1,2,3\})$.

To decide between the two possibilities is quite easy: we compute $(p^i)^{(id,id,(13))}$ and, applying Lemma 1 iii), we find the unique p^j such that $(p^i)^{(id,id,(13))} = (p^j)^{(\varphi,\psi,id)}$, by suitable $\psi \in S_3, \varphi \in S_5$. Note that, due to our procedure, we can limit the research to the indices $j \geq i$ and avoid the eliminated components of P. If $j = i$, then we are in the first situation; if $j \neq i$, then we are in the second situation.

To give the flavor of our method we examine two expressive cases in detail.

Let $J = \{1, \dots, 42\}$ and consider first

$$p^{28} = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 1 \\ 3 & 3 & 1 & 1 & 3 \end{bmatrix}$$

so that

$$(p^{28})^{(id,id,(13))} = \begin{bmatrix} 3 & 3 & 1 & 1 & 3 \\ 2 & 2 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

We want to find the unique $j \in J$ for which there exists $\psi \in S_3, \varphi \in S_5$ such that

$$(p^{28})^{(id,id,(13))} = (p^j)^{(\varphi,\psi,id)}. \tag{22}$$

Note that only one such ψ is possible: namely if we have also $(p^{28})^{(id,id,(13))} = (p^j)^{(\hat{\varphi},\hat{\psi},id)}$, then $(p^j)^{(\hat{\varphi}^{-1}\varphi,\hat{\psi}^{-1}\psi,id)} = p^j$ and thus $(\hat{\varphi}^{-1}\varphi, \hat{\psi}^{-1}\psi, id) \in \mathrm{Stab}_U(p^j)$ so that, by Lemma 8 ii), we deduce $\hat{\psi}^{-1}\psi = id$.

We write equation (22) equivalently as

$$(p^{28})^{(id,\psi^{-1},(13))} = (p^j)^{(\varphi,id,id)}, \tag{23}$$

so that our task reduces to find the unique $j \in J$ and the unique $\psi \in S_3$, such that the profile

$$(p^{28})^{(id,\psi^{-1},(13))} = \begin{bmatrix} 3 & 3 & 1 & 1 & 3 \\ 2 & 2 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix}^{\psi^{-1}},$$

up to a permutations of the columns, coincides with p^j.

Since, by their definition, in each p^j the maximum number of equal columns is realized by $[1,2,3]^T$, our p^j must have two columns equal to $[1,2,3]^T$ and admit other two equal columns [12]. Thus ψ^{-1} must map $[3,2,1]^T$ into $[1,2,3]^T$ or $[1,3,2]^T$ into $[1,2,3]^T$. This gives $\psi^{-1} \in \{(13), (23)\}$. Since it is immediate to check that the choice $\psi^{-1} = (23)$ gives $(p^{28})^{(id,(23),(13))} = (p^{28})^{(\varphi,id,id)}$ for a suitable $\varphi \in S_5$, we find $j = 28, (p^{28})^G = (p^{28})^U$ and $S^0(p^{28}) = \{q_0 \in S_3 : (23)q_0(13) = q_0\}$. But the equation $(23)q_0(13) = q_0$ is equivalent to $(23) = q_0(13)q_0^{-1}$, that is to $(23) = (q_0(1)\ q_0(3))$, which says that we must have $q_0(1) = 2$ and $q_0(3) = 3$ or $q_0(1) = 3$ and $q_0(3) = 2$. Thus we find

$$S^0(p^{28}) = \{q_0 \in S_3 : (23)q_0(13) = q_0\} = \{[2,1,3]^T, [3,1,2]^T\}.$$

[12]In other words the block type of p^j is $\mathcal{P}(2,2,1)$ (see Bubboloni and Gori (2013))

Note that p^{28} is q^{17} in the list Q.

Consider now the different behavior of

$$p^{34} = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 1 & 2 \end{bmatrix}.$$

We have

$$(p^{34})^{(id,id,(13))} = \begin{bmatrix} 3 & 3 & 3 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 & 3 \end{bmatrix}$$

and, as in the previous case, we look for the unique $j \in J$ and the unique $\psi \in S_3$, such that the profile

$$(p^{34})^{(id,\psi^{-1},(13))} = \begin{bmatrix} 3 & 3 & 3 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 & 3 \end{bmatrix}^{\psi^{-1}}$$

up to a permutations of the columns, coincides with p^j. Here for $\psi = (13)$ and $\varphi = (34)$, we get $(p^{34})^{((34),(13),(13))} = p^{38}$. Thus $j = 38$, $(p^{34})^G = (p^{34})^U \cup (p^{38})^U$ and $S^0(p^{34}) = \mathcal{L}(\{1,2,3\}) = S_3$. Note that p^{34} is q^{22} in the list Q, while p^{38} does not appear any more.

Once the list Q is complete we get, in particular, $R(G) = 26$ just counting the length of the list Q. Moreover the knowledge of $S^0(q^i)$ together with the knowledge of the sets $C_\nu(q^i)$ enable us to get, by intersection, also $S^1(q^i)$. After that the order of all the types of rules is immediate.

5 Concluding comments

The strong arithmetical condition $\gcd(h, n!) = 1$ can be simply recognized as a necessary condition to have anonymous and neutral rules. What we point out in this paper is that such a condition is indeed sufficient for the existence of rules not only satisfying anonymity and neutrality but also many other interesting properties like reversal symmetry and minimal majority. As a consequence, each time h individuals want to strict rank n alternatives aggregating their preferences and obeying to the principles of anonymity, neutrality, reversal symmetry and minimal majority, they know that there is an aggregation procedure having the desired properties if and only if $\gcd(h, n!) = 1$.

Further, they also have a practical method to (potentially) build all the procedures of that type. For instance, making computations by hands, we showed in the previous section that when individuals are five and alternatives are three such procedures are two. Moreover, using the list of representatives $q^1, \ldots, q^{26}$ and the table describing the sets S^1, they both can be easily implemented on a calculator. Thus, individuals have only to select one of the two procedures on the basis of some other shared criteria. In other words, they have to find an agreement about which social outcome between $[1,2,3]^T$ and $[3,2,1]^T$ has to be associated with the preference profile

$$q^{25} = \begin{bmatrix} 1 & 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 1 & 1 & 2 \end{bmatrix}$$

A possible way out could be to use the further principle stating that between two possible social outcomes both consistent with anonymity, neutrality, reversal symmetry and minimal majority it should be preferred the one mostly expressed by individuals, if any. If individuals also agree on this principle, then there is only one rule supporting their requirements as only $[1,2,3]^T$ can be associated with q^{25}, appearing twice in the profile.

We think that, according to these observations, an interesting research project could be to find out, under the assumption $\gcd(h, n!) = 1$, further reasonable principles to be added to the ones considered in the paper that lead to select a unique rule.

6 Appendix

In this section we give the proof of Proposition 16. The proof is tricky and requires, to be developed, a deep insight into the relation

$$\Sigma_\nu(p) = \Big\{(x, y) \in N \times N : |\{i \in H : x >_{p_i} y\}| \geq \nu\Big\},$$

which we are going to consider for $p \in \mathcal{P}$, with $\mathrm{Stab}_G(p) > \mathrm{Stab}_U(p)$, and $\nu = \nu(p)$. We will need five lemmas concerning the relation $\Sigma_\nu(p)$ to get all the tools for the proof.

6.1 The relation $\Sigma_\nu(p)$

Throughout this section assume $\gcd(h, n!) = 1$ *and* $\nu = \nu(p)$*, fix* $p \in \mathcal{P}$ *with* $\mathrm{Stab}_G(p) > \mathrm{Stab}_U(p)$ *and* $(\varphi, \psi, \rho_0) \in \mathrm{Stab}_G(p)$.

We start our study of $\Sigma_\nu(p)$ observing that since $C_\nu(p)$, the set of the linear extensions of $\Sigma_\nu(p)$, is non-empty, we know that $\Sigma_\nu(p)$ is acyclic. Note also that $\Sigma_\nu(p)$ is asymmetric and generally not transitive and not complete. In particular $\Sigma_\nu(p)$ is irreflexive, that is if for some $x, y \in N$ we have $(x, y) \in \Sigma_\nu(p)$, then $x \neq y$. We will write $x >_\nu y$ instead of $(x, y) \in \Sigma_\nu(p)$.

Our first result is about the role of ψ in the relation $\Sigma_\nu(p)$.

Lemma 20. *Let* $x, y \in N$*. Then* $x >_\nu y$ *if and only if* $\psi(y) >_\nu \psi(x)$*.*

Proof. Let $x, y \in N$ and consider the two subsets of H given by $A = \{i \in H : x >_{p_i} y\}$ and $B = \{i \in H : \psi(y) >_{p_i} \psi(x)\}$. Clearly it is enough to show $|A| = |B|$. We prove $\varphi(A) \subseteq B$ and $\varphi(B) \subseteq A$. If $i \in A$, then we have $x >_{p_i} y$ and thus, by the relations (10) and (11), we get $\psi(y) >_{\psi p_i \rho_0} \psi(x)$. But since $(\varphi, \psi, \rho_0) \in \mathrm{Stab}_G(p)$ we have that $\psi p_i \rho_0 = p_{\varphi(i)}$ and thus $\psi(y) >_{p_{\varphi(i)}} \psi(x)$, that is $\varphi(i) \in B$. Next let $i \in B$, that is $\psi(y) >_{p_i} \psi(x)$; by what proved above it follows that $x = \psi\psi(x) >_{p_{\varphi(i)}} \psi\psi(y) = y$, which means $\varphi(i) \in A$. □

Consider now the following relation over N,

$$\Sigma^C_\nu(p) = \{(x, y) \in N \times N : \text{there exists a } \Sigma_\nu(p)\text{-chain from } x \text{ to } y\},$$

and note that $\Sigma^C_\nu(p) \supseteq \Sigma_\nu(p)$.

Lemma 21. $\Sigma^C_\nu(p)$ *is asymmetric and transitive. Moreover,* $(x, y) \in \Sigma^C_\nu(p)$ *if and only if* $(\psi(y), \psi(x)) \in \Sigma^C_\nu(p)$*.*

Proof. Let us prove first that $\Sigma^C_\nu(p)$ is asymmetric. Let $(x, y) \in \Sigma^C_\nu(p)$. Then, there exists a $\Sigma_\nu(p)$-chain from x to y, that is there exist $l \geq 2$ distinct $x_1 \ldots, x_l \in N$ such that $x = x_1$, $y = x_l$ and, for every $j \in \{1, \ldots, l-1\}$, $x_j >_\nu x_{j+1}$. In particular, we have $x \neq y$. Assume, by contradiction, that $(y, x) \in \Sigma^C_\nu(p)$. Then there exist $k \geq 2$ distinct $y_1 \ldots, y_k \in N$ such that $y = y_1$, $x = y_k$ and, for every $j \in \{1, \ldots, k-1\}$, $y_j >_\nu y_{j+1}$. Consider now the set

$$A = \{j \in \{2, \ldots, k\} : y_j \in \{x_1, \ldots, x_{l-1}\}\}.$$

Clearly, being $y_k = x_1$, we have $k \in A \neq \varnothing$. Let us define then $k^* = \min A$, so that there exists $l^* \in \{1, \ldots, l-1\}$ such that $y_{k^*} = x_{l^*}$. Then, recalling that $x \neq y$, we easily check that $x_{l^*}, x_{l^*+1}, \ldots, x_l, y_2, \ldots, y_{k^*}$ is a sequence of at least three elements in N, with no repetition up to the $x_{l^*} = y_{k^*}$, which is a cycle in $\Sigma_\nu(p)$ and the contradiction is found.

Let us prove now that $\Sigma^C_\nu(p)$ is transitive. Let $(x, y), (y, z) \in \Sigma^C_\nu(p)$. Then, by definition of $\Sigma^C_\nu(p)$, there exist $l \geq 2$ distinct $x_1 \ldots, x_l \in N$ such that $x = x_1$, $y = x_l$ and, for every $j \in \{1, \ldots, l-1\}$, $x_j >_\nu x_{j+1}$; moreover there are $k \geq 2$ distinct $y_1 \ldots, y_k \in N$ such that $y = y_1$,

$z = y_k$ and, for every $j \in \{1, \dots, k-1\}$, $y_j >_\nu y_{j+1}$. Consider then the sequence of alternatives $x_1, \dots, x_l, y_2, \dots, y_k$. As $\Sigma_\nu(p)$ is acyclic those alternatives are all distinct and construct a chain from x to z, so that $(x, z) \in \Sigma_\nu^C(p)$.

We are left with proving $(x, y) \in \Sigma_\nu^C(p)$ if and only if $(\psi(y), \psi(x)) \in \Sigma_\nu^C(p)$. Let $(x, y) \in \Sigma_\nu^C(p)$ and consider $l \geq 2$ distinct $x_1 \dots, x_l \in N$ such that $x = x_1$, $y = x_l$ and, for every $j \in \{1, \dots, l-1\}$, $x_j >_\nu x_{j+1}$. Defining, for every $j \in \{1, \dots, l\}$, $y_j = \psi(x_{l-j+1})$ and using Lemma 20, it is immediately checked that $\psi(y) = y_1$, $\psi(x) = y_l$ and that, for every $j \in \{1, \dots, l-1\}$, $y_j >_\nu y_{j+1}$. In other words, we have a $\Sigma_\nu^C(p)$-chain from $\psi(y)$ to $\psi(x)$, that is $(\psi(y), \psi(x)) \in \Sigma_\nu^C(p)$. The other implication is now a trivial consequence of $|\psi| = 2$. ☐

In what follows, we write $x \hookrightarrow_\nu y$ instead of $(x, y) \in \Sigma_\nu^C(p)$. Thus Lemma 21 may be rephrased saying that, for every $x, y, z \in N$:

- $x \not\hookrightarrow_\nu x$,
- $x \hookrightarrow_\nu y$ implies $y \not\hookrightarrow_\nu x$,
- $x \hookrightarrow_\nu y$ and $y \hookrightarrow_\nu z$ imply $x \hookrightarrow_\nu z$,
- $x \hookrightarrow_\nu y$ is equivalent to $\psi(y) \hookrightarrow_\nu \psi(x)$.

Define now, for every $z \in N$, the following subset of N:

$$\Gamma(z) = \{x \in N : x \hookrightarrow_\nu z\}$$

Note that $z \notin \Gamma(z)$ and that it may happen that $\Gamma(z) = \varnothing$: this is the case exactly when for all $x \in N$ the relation $x >_\nu z$ does not hold. Define now the subset of N:

$$\Gamma = \bigcup_{z \in N} [\Gamma(z) \cap \Gamma(\psi(z))].$$

This set represent those alternatives $x \in N$ from which we can reach, by a $\Sigma_\nu(p)$-chain, both z and $\psi(z)$, for some $z \in N$.

Our idea is that Γ collects those alternatives which necessarily belong to the superior half part of each element in $S^1(p)$, because the majority relations implied by p oblige them. Symmetrically $\psi(\Gamma)$ collects those alternatives which necessarily belongs to the inferior half part of each element in $S^1(p)$.

To make explicit this fact we proceed by steps. First we show that each alternative which is over some alternative in Γ is itself in Γ.

Lemma 22. *Let $x, y \in N$. If $y \in \Gamma$ and $x \hookrightarrow_\nu y$ then $x \in \Gamma$.*

Proof. Let $x, y \in N$ and $x \hookrightarrow_\nu y$ with $y \in \Gamma$. Thus there exists $z \in N$ such that $y \hookrightarrow_\nu z$ and $y \hookrightarrow_\nu \psi(z)$. By transitivity we conclude that also $x \hookrightarrow_\nu z$ and $x \hookrightarrow_\nu \psi(z)$, that is, $x \in \Gamma$. ☐

Then we study the set Γ with respect to ψ. Recall that, by Proposition 9 b)ii), ψ has a fixed point if and only if n is odd and that this fixed point is unique. This will lead to the awareness that Γ contains at most the half of the alternatives and that in no case the fixed point belongs to Γ.

Lemma 23. *i)* $\Gamma \cap \psi(\Gamma) = \varnothing$;

ii) Let $x_0 \in N$ be a fixed point for ψ. Then $x_0 \notin \Gamma \cup \psi(\Gamma)$. Moreover if $x \in \Gamma$, then $x_0 \not>_\nu x$;

iii) For every $x \in N$, $|\{x, \psi(x)\} \cap \Gamma| \leq 1$;

iv) $0 \leq |\Gamma| \leq \lfloor \frac{n}{2} \rfloor$.

Proof. i) Assume that there exists $x \in N$ with $x \in \Gamma$ and $x \in \psi(\Gamma)$, that is $\psi(x) \in \Gamma$. Then there exist $z, y \in N$ with

$$x \hookrightarrow_\nu z, \quad x \hookrightarrow_\nu \psi(z), \quad \psi(x) \hookrightarrow_\nu y, \quad \psi(x) \hookrightarrow_\nu \psi(y). \tag{24}$$

By Lemma 21 applied to the relations in (24), we get also

$$\psi(z) \hookrightarrow_\nu \psi(x), \quad z \hookrightarrow_\nu \psi(x), \quad \psi(y) \hookrightarrow_\nu x, \quad y \hookrightarrow_\nu x. \tag{25}$$

From (24) and (25) and by Lemma 23, we deduce that $\psi(x) \hookrightarrow_\nu x$ and $x \hookrightarrow_\nu \psi(x)$, against the asymmetry of $\Sigma_\nu^C(p)$ established in Lemma 21.

ii) Assume that $x_0 \in \Gamma \cup \psi(\Gamma)$. Then, by i), we have $x_0 = \psi(x_0) \in \Gamma \cap \psi(\Gamma) = \varnothing$, a contradiction. Next let $x \in \Gamma$ and $x_0 >_\nu x$. Then, by Lemma 22, we have also $x_0 \in \Gamma$, a contradiction.

iii) Assume there is $x \in N$ such that both x and $\psi(x)$ belong to Γ. Then $x \in \psi(\Gamma)$ and by i), $x \in \Gamma \cap \psi(\Gamma) = \varnothing$, a contradiction.

iv) By i), we have that the sets Γ and $\psi(\Gamma)$ are disjoint sets of the same size included in N. Thus, if n is even we have $|\Gamma| \leq \frac{n}{2}$. If n is odd, using also ii, we have that $\Gamma \subseteq N \setminus \{x_0\}$ and so $|\Gamma| \leq \frac{n-1}{2}$. In both cases we therefore have $|\Gamma| \leq \lfloor \frac{n}{2} \rfloor$. □

Finally we explain how to decide which among $x \in N$ and $\psi(x)$ belongs to Γ, looking at the behavior with respect to the fixed point x_0 of ψ, in the odd case. If no information of the type $x >_\nu x_0$ or $x_0 >_\nu x$ is available, we will have in $S^1(p)$ orders in which x is in the superior half part and others in which x is in the inferior half part.

Lemma 24. *Let $x_0 \in N$ be a fixed point of ψ and $x \in N$.*

i) If $x >_\nu x_0$ then $\{x, \psi(x)\} \cap \Gamma = \{x\}$.

ii) If $x_0 >_\nu x$ then $\{x, \psi(x)\} \cap \Gamma = \{\psi(x)\}$.

Proof. i) From $x >_\nu x_0$, it follows trivially also that $x >_\nu \psi(x_0)$ and thus $x \in \Gamma(x_0) \cap \Gamma(\psi(x_0)) \subseteq \Gamma$. By Lemma 23 i) it cannot be also $\psi(x) \in \Gamma$ and thus $\{x, \psi(x)\} \cap \Gamma = \{x\}$.

ii) From $x_0 >_\nu x$, using Lemma 20, we obtain $\psi(x) >_\nu x_0$ and i) applies to $\psi(x)$, giving $\{x, \psi(x)\} \cap \Gamma = \{\psi(x)\}$.

□

6.2 Proof of Proposition 16

Let $\gcd(h, n!) = 1$ and $p \in \mathcal{P}$. We want to show that $S^1(p) = S^0(p) \cap C_{\nu(p)}(p) \neq \varnothing$. If $\mathrm{Stab}_G(p) = \mathrm{Stab}_U(p)$, then $S^0(p) = \mathcal{L}(N)$ and so $S^1(p) = C_{\nu(p)}(p) \neq \varnothing$. Assume now $\mathrm{Stab}_G(p) > \mathrm{Stab}_U(p)$ and, for simplicity in notation, put $\nu = \nu(p)$. Let $(\varphi, \psi, \rho_0) \in \mathrm{Stab}_G(p)$ and observe that the assumptions of Section 6.1 are satisfied.

Given $X \subseteq N$, define

$$\mathcal{C}_\nu(p, X) = \Big\{ f \in \mathcal{L}(X) : f \supseteq \Sigma_\nu(p) \cap (X \times X) \Big\}$$

that is, the set of the linear extensions of $\Sigma_\nu(p)$ restricted to X. Note that $\mathcal{C}_\nu(p, N) = \mathcal{C}_\nu(p)$. Note also that if $Y \subseteq X \subseteq N$ and $f \in \mathcal{C}_\nu(p, X)$, then the restriction of f to Y belongs to $\mathcal{C}_\nu(p, Y)$.

By Proposition 9 b)ii), there exist $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in N$ such that

$$\Big\{ \{x_i, \psi(x_i)\} : i \in \{1, \ldots, k\} \Big\}$$

is a partition of N. Moreover if n is even, then $k = \frac{n}{2}$ and, for every $i \in \{1, \ldots, k\}$, $|\{x_i, \psi(x_i)\}| = 2$; if n is odd, then $k = \frac{n+1}{2}$ and we can assume that, for every $i \in \{1, \ldots, k-1\}$, $|\{x_i, \psi(x_i)\}| = 2$ while x_k is the unique fixed point of ψ. Define $K = \{1, \ldots, k\}$.

For every $i \in K$, consider $\{x_i, \psi(x_i)\} \cap \Gamma$: by Lemma 23 iii) the order of this set cannot exceed 1. Define then the sets

$$J = \{i \in K : |\{x_i, \psi(x_i)\} \cap \Gamma| = 1\}, \quad J^* = \{i \in K \setminus J : |\{x_i, \psi(x_i)\}| = 2\}.$$

Of course, for every $i \in K \setminus J$, $\{x_i, \psi(x_i)\} \cap \Gamma = \varnothing$. Note also that, when n is even, we have $J \cup J^* = K$; if n is odd, by Lemma 23 ii), we have $J \cup J^* = K \setminus \{k\}$. For every $i \in J$, let us call y_i the unique element in the set $\{x_i, \psi(x_i)\} \cap \Gamma$ so that $\Gamma = \{y_i : i \in J\}$.

Consider now the subset of N defined by

$$T = \{y_i : i \in J\} \cup \bigcup_{i \in J^*} \{x_i, \psi(x_i)\}$$

and note that $T \supseteq \Gamma$. We consider the relation $\Sigma_\nu(p) \cap (T \times T)$. That relation is acyclic, because included in the acyclic $\Sigma_\nu(p)$, and thus we have $C_\nu(p, T) \neq \varnothing$. Let $f \in C_\nu(p, T)$ and for $i \in J^*$, let y_i be the maximum of $\{x_i, \psi(x_i)\}$ with respect to f, so that $y_i >_f \psi(y_i)$. Define

$$U = \{y_i : i \in J \cup J^*\}.$$

Observe that if n is even, then $U \cup \psi(U) = N$ and if n is odd, then $U \cup \psi(U) = N \setminus \{x_k\}$. Moreover, for all $n \in \mathbb{N}$ with $n \geq 2$, we have $|U| = \lfloor \frac{n}{2} \rfloor$, $U \cap \psi(U) = \varnothing$ and $\Gamma \subseteq U \subseteq T$. The restriction of $f \in C_\nu(p, T)$ to U is a linear order $g \in C_\nu(p, U)$, say

$$g = \left[a_1, \ldots, a_{\lfloor \frac{n}{2} \rfloor}\right]^T$$

with $U = \{a_1, \ldots, a_{\lfloor \frac{n}{2} \rfloor}\}$. Now consider the linear order over N given by

$$q = \begin{cases} \left[a_1, \ldots, a_{\lfloor \frac{n}{2} \rfloor}, \psi\left(a_{\lfloor \frac{n}{2} \rfloor}\right), \ldots, \ldots, \psi(a_1)\right]^T & \text{if } n \text{ is even} \\ \\ \left[a_1, \ldots, a_{\lfloor \frac{n}{2} \rfloor}, x_k, \psi\left(a_{\lfloor \frac{n}{2} \rfloor}\right), \ldots, \ldots, \psi(a_1)\right]^T & \text{if } n \text{ is odd} \end{cases}$$

Note that in the upper part of the list appear the alternatives in U and in the lower one those in $\psi(U)$; in the odd case, the fixed point x_k of ψ is ranked in the middle, to the position $k = \frac{n+1}{2}$. By construction, we have that $\psi q(n - i + 1) = q(i)$ for all $i \in N$, that is $\psi q \rho_0 = q$, which means that $q \in S^0(p)$. As a consequence we have that

$$\forall\, x, y \in N,\ y >_q x \text{ is equivalent to } \psi(x) >_q \psi(y). \tag{26}$$

Namely, by (10) and (11), we have

$$y >_q x \quad \Leftrightarrow \quad y >_{\psi q \rho_0} x \quad \Leftrightarrow \quad x >_{\psi q} y \quad \Leftrightarrow \quad \psi(x) >_q \psi(y.)$$

In order to complete the proof we need to show that $q \in C_\nu(p)$, that is that for all $x, y \in N$ with $x >_\nu y$, we have $x >_q y$. Since when n is even we have $N = U \cup \psi(U)$ and when n is odd we have $N = U \cup \psi(U) \cup \{x_k\}$, we reduce to prove that, for every $x, y \in U$:

a) $x >_\nu y$ implies $x >_q y$,

b) $x >_\nu \psi(y)$ implies $x >_q \psi(y)$,

c) $\psi(x) >_\nu \psi(y)$ implies $\psi(x) >_q \psi(y)$,

d) $\psi(x) >_\nu y$ implies $\psi(x) >_q y$, and, in the odd case, showing further that, for every $x \in U$:

e) $x >_\nu x_k$ implies $x >_q x_k$,

f) $x_k >_\nu x$ implies $x_k >_q x$,

g) $\psi(x) >_\nu x_k$ implies $\psi(x) >_q x_k$,

h) $x_k >_\nu \psi(x)$ implies $x_k >_q \psi(x)$.

Fix then $x, y \in U$ and prove the points a), b), c) and d). Note that $x \neq \psi(x)$ and $y \neq \psi(y)$, because we have observed that if ψ admits a fixed point x_k, then $x_k \notin U$.

a) Let $x >_\nu y$. Since $g \in C_\nu(p, U)$, we have $x >_g y$ and then also $x >_q y$.

b) By the construction of q, for all $x, y \in U$, we have $x >_q \psi(y)$.

c) Let $\psi(x) >_\nu \psi(y)$. Then, by Lemma 20, we have also $y >_\nu x$ and by a) we get $y >_q x$, which by (26) implies $\psi(x) >_q \psi(y)$.

d) Let us prove that it cannot be $\psi(x) >_\nu y$. Assume, by contradiction, $\psi(x) >_\nu y$. Note that, by Lemma 20 we also have $\psi(y) >_\nu x$. If $y \in \Gamma$, then, by Lemma 22, we have $\psi(x) \in \Gamma \subseteq U$ and thus $x \in U \cap \psi(U) = \varnothing$, a contradiction. Assume now that $y \notin \Gamma$. If $x \in \Gamma$, then $\psi(y) \in \Gamma \subseteq U$ and we get the contradiction $y \in U \cap \psi(U) = \varnothing$. If instead $x \notin \Gamma$, then $x, \psi(x), y, \psi(y) \in T$. As a consequence, $\psi(x) >_\nu y$ implies $\psi(x) >_f y$ and $\psi(y) >_\nu x$ implies $\psi(y) >_f x$. Moreover, as y is the maximum of $\{y, \psi(y)\}$ with respect to f we have $y >_f \psi(y)$. By transitivity of f, we then get $\psi(x) >_f x$, which is a contradiction because x is the maximum of $\{x, \psi(x)\}$ with respect to f .

Assume now that n is odd. Fix then $x \in U$ and prove the points e), f), g) and h).

e) This case is trivial, because for each $x \in U$ we have $x >_q x_k$.

f) Let us prove that it cannot be $x_k >_\nu x$. Assume, by contradiction, $x_k >_\nu x$. Then, by Lemma 20 we also have $\psi(x) >_\nu x_k$, which by Lemma 24 i) gives $\psi(x) \in \Gamma \subseteq U$, against $U \cap \psi(U) = \varnothing$.

g) Let us prove that it cannot be $\psi(x) >_\nu x_k$. Assume, by contradiction, $\psi(x) >_\nu x_k$. Then, by Lemma 24 i), we get $\psi(x) \in \Gamma \subseteq U$ against $U \cap \psi(U) = \varnothing$.

h) This case is trivial because, by the construction of q, we have $x_k >_q \psi(x)$ for all $x \in U$.

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