DISCRETE MATHEMATICS

W W L CHEN

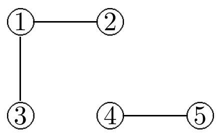
Chapter 17

GRAPHS

17.1. Introduction

A graph is simply a collection of vertices, together with some edges joining some of these vertices.

Example 17.1.1. The graph



has vertices $1, 2, 3, 4, 5$, while the edges may be described by $\{1, 2\}, \{1, 3\}, \{4, 5\}$. In particular, any edge can be described as a 2-subset of the set of all vertices; in other words, a subset of 2 elements of the set of all vertices.

Definition. A graph is an object $G = (V, E)$, where V is a finite set and E is a collection of 2-subsets of V. The elements of V are known as vertices and the elements of E are known as edges. Two vertices $x, y \in V$ are said to be adjacent if $\{x, y\} \in E$; in other words, if x and y are joined by an edge.

Example 17.1.2. In our earlier example, $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}$. The vertices 2 and 3 are both adjacent to the vertex 1, but are not adjacent to each other since $\{2, 3\} \notin E$. We can also represent this graph by an adjacency list

1	2	3	4	5
2	1	1	5	4
3				

where each vertex heads a list of those vertices adjacent to it.

Remark. Note that our definition does not permit any edge to join the same vertex, so that there are no "loops". Note also that the edges do not have directions.

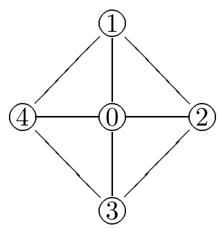
Example 17.1.3. For every $n \in \mathbb{N}$, the wheel graph $W_n = (V, E)$, where $V = \{0, 1, 2, \ldots, n\}$ and

$$E = \{\{0,1\}, \{0,2\}, \ldots, \{0,n\}, \{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\}.$$

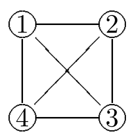
We can represent this graph by the adjacency list below.

0	1	2	3	...	$n-1$	n
1	0	0	0	...	0	0
⋮	2	3	4	...	n	1
n	n	1	2	...	$n-2$	$n-1$

For example, W_4 can be illustrated by the picture below.



Example 17.1.4. For every $n \in \mathbb{N}$, the complete graph $K_n = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, j\} : 1 \le i < j \le n\}$. In this graph, every pair of distinct vertices are adjacent. For example, K_4 can be illustrated by the picture below.

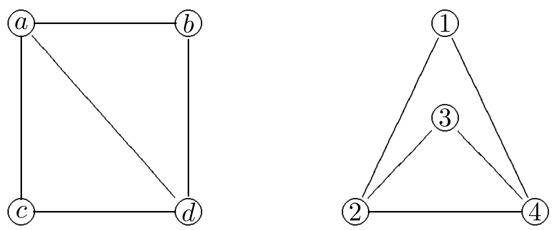


In Example 17.1.4, we called K_n the complete graph. This calls into question the situation when we may have another graph with n vertices and where every pair of dictinct vertices are adjacent. We have then to accept that the two graphs are essentially the same.

Definition. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a function $\alpha : V_1 \to V_2$ which satisfies the following properties:

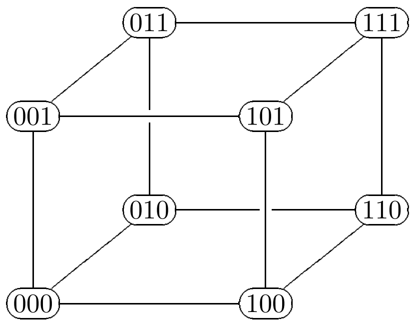
(GI1) $\alpha : V_1 \to V_2$ is one-to-one.
(GI2) $\alpha : V_1 \to V_2$ is onto.
(GI3) For every $x, y \in V_1$, $\{x, y\} \in E_1$ if and only if $\{\alpha(x), \alpha(y)\} \in E_2$.

Example 17.1.5. The two graphs below are isomorphic.



Simply let, for example, $\alpha(a) = 2$, $\alpha(d) = 4$, $\alpha(b) = 1$ and $\alpha(c) = 3$.

EXAMPLE 17.1.6. Let $G = (V, E)$ be defined as follows. Let V be the collection of all strings of length 3 in $\{0, 1\}$. In other words, $V = \{x_1x_2x_3 : x_1, x_2, x_3 \in \{0, 1\}\}$. Let E contain precisely those pairs of strings which differ in exactly one position, so that, for example, $\{000, 010\} \in E$ while $\{101, 110\} \notin E$. We can show that G is isomorphic to the graph formed by the corners and edges of an ordinary cube. This is best demonstrated by the following pictorial representation of G.



17.2. Valency

DEFINITION. Suppose that $G = (V, E)$, and let $v \in V$ be a vertex of G. By the valency of v, we mean the number

$$\delta(v) = |\{e \in E : v \in e\}|,$$

the number of edges of G which contain the vertex v.

EXAMPLE 17.2.1. For the wheel graph W_4, $\delta(0) = 4$ and $\delta(v) = 3$ for every $v \in \{1, 2, 3, 4\}$.

EXAMPLE 17.2.2. For every complete graph K_n, $\delta(v) = n - 1$ for every $v \in V$.

Note that each edge contains two vertices, so we immediately have the following simple result.

PROPOSITION 17A. *The sum of the values of the valency $\delta(v)$, taken over all vertices v of a graph $G = (V, E)$, is equal to twice the number of edges of G. In other words,*

$$\sum_{v \in V} \delta(v) = 2|E|.$$

PROOF. Consider an edge $\{x, y\}$. It will contribute 1 to each of the values $\delta(x)$ and $\delta(y)$ and $|E|$. The result follows on adding together the contributions from all the edges. ◯

In fact, we can say a bit more.

DEFINITION. A vertex of a graph $G = (V, E)$ is said to be odd (resp. even) if its valency $\delta(v)$ is odd (resp. even).

PROPOSITION 17B. *The number of odd vertices of a graph $G = (V, E)$ is even.*

PROOF. Let V_e and V_o denote respectively the collection of even and odd vertices of G. Then it follows from Proposition 17A that

$$\sum_{v \in V_e} \delta(v) + \sum_{v \in V_o} \delta(v) = 2|E|.$$

For every $v \in V_e$, the valency $\delta(v)$ is even. It follows that

$$\sum_{v \in V_o} \delta(v)$$

is even. Since $\delta(v)$ is odd for every $v \in V_o$, it follows that $|V_o|$ must be even. ◯

EXAMPLE 17.2.3. For every $n \in \mathbb{N}$ satisfying $n \geq 3$, the cycle graph $C_n = (V, E)$, where $V = \{1, \ldots, n\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. Here every vertex has velancy 2.

EXAMPLE 17.2.4. A graph $G = (V, E)$ is said to be regular with valency r if $\delta(v) = r$ for every $v \in V$. In particular, the complete graph K_n is regular with valency $n - 1$ for every $n \in \mathbb{N}$.

REMARK. The notion of valency can be used to test whether two graphs are isomorphic. It is not difficult to see that if $\alpha : V_1 \to V_2$ gives an isomorphism between graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then we must have $\delta(v) = \delta(\alpha(v))$ for every $v \in V_1$.

17.3. Walks, Paths and Cycles

Graph theory is particularly useful for routing purposes. After all, a map can be thought of as a graph, with places as vertices and roads as edges (here we assume that there are no one-way streets). It is therefore not unusual for some terms in graph theory to have some very practical-sounding names.

DEFINITION. A walk in a graph $G = (V, E)$ is a sequence of vertices

$$v_0, v_1, \ldots, v_k \in V$$

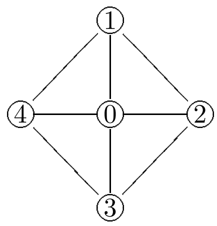
such that for every $i = 1, \ldots, k$, $\{v_{i-1}, v_i\} \in E$. In this case, we say that the walk is from v_0 to v_k. Furthermore, if all the vertices are distinct, then the walk is called a path. On the other hand, if all the vertices are distinct except that $v_0 = v_k$, then the walk is called a cycle.

REMARK. A walk can also be thought of as a succession of edges

$$\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}.$$

Note that a walk may visit any given vertex more than once or use any given edge more than once.

EXAMPLE 17.3.1. Consider the wheel graph W_4 described in the picture below.



Then $0, 1, 2, 0, 3, 4, 3$ is a walk but not a path or cycle. The walk $0, 1, 2, 3, 4$ is a path, while the walk $0, 1, 2, 0$ is a cycle.

Suppose that $G = (V, E)$ is a graph. Define a relation $\sim$ on V in the following way. Suppose that $x, y \in V$. Then we write $x \sim y$ whenever there exists a walk

$$v_0, v_1, \ldots, v_k \in V$$

with $x = v_0$ and $y = v_k$. Then it is not difficult to check that $\sim$ is an equivalence relation on V. Let

$$V = V_1 \cup \ldots \cup V_r,$$

a (disjoint) union of the distinct equivalence classes. For every $i = 1, \ldots, r$, let

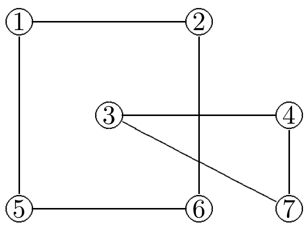
$$E_i = \{\{x, y\} \in E : x, y \in V_i\};$$

in other words, E_i denotes the collection of all edges in E with both endpoints in V_i. It is not hard to see that $E_1, \ldots, E_r$ are pairwise disjoint.

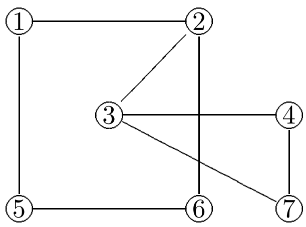
Definition. For every $i = 1, \ldots, r$, the graphs $G_i = (V_i, E_i)$, where V_i and E_i are defined above, are called the components of G. If G has just one component, then we say that G is connected.

Remark. A graph $G = (V, E)$ is connected if for every pair of distinct vertices $x, y \in V$, there exists a walk from x to y.

Example 17.3.2. The graph described by the picture



has two components, while the graph described by the picture



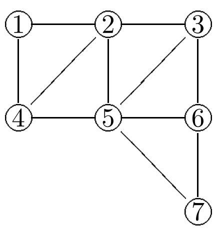
is connected.

Example 17.3.3. For every $n \in \mathbb{N}$, the complete graph K_n is connected.

Sometimes, it is rather difficult to decide whether or not a given graph is connected. We shall develop some algorithms later to study this problem.

17.4. Hamiltonian Cycles and Eulerian Walks

At some imaginary time, the mathematicians Hamilton and Euler went for a holiday. They visited a country with 7 cities (vertices) linked by a system of roads (edges) described by the following graph.



Hamilton is a great mathematician. He would only be satisfied if he could visit each city once and return to his starting city. Euler is an immortal mathematician. He was interested in the scenery on the way as well and would only be satisfied if he could follow each road exactly once, and would not mind ending his trip in a city different from where he started.

Hamilton was satisfied, but Euler was not.

To see that Hamilton was satisfied, note that he could follow, for example, the cycle

$$1, 2, 3, 6, 7, 5, 4, 1.$$

However, to see that Euler was not satisfied, we need to study the problem a little further. Suppose that Euler attempted to start at x and finish at y. Let z be a vertex different from x and y. Whenever Euler arrived at z, he needed to leave via a road he had not taken before. Hence z must be an even vertex. Furthermore, if $x \neq y$, then both vertices x and y must be odd; if $x = y$, then both vertices x and y must be even. It follows that for Euler to succeed, there could be at most two odd vertices. Note now that the vertices $1, 2, 3, 4, 5, 6, 7$ have valencies $2, 4, 3, 3, 5, 3, 2$ respectively!

DEFINITION. A hamiltonian cycle in a graph $G = (V, E)$ is a cycle which contains all the vertices of V.

DEFINITION. An eulerian walk in a graph $G = (V, E)$ is a walk which uses each edge in E exactly once.

We shall state without proof the following result.

PROPOSITION 17C. *In a graph $G = (V, E)$, a necessary and sufficient condition for an eulerian walk to exist is that G has at most two odd vertices.*

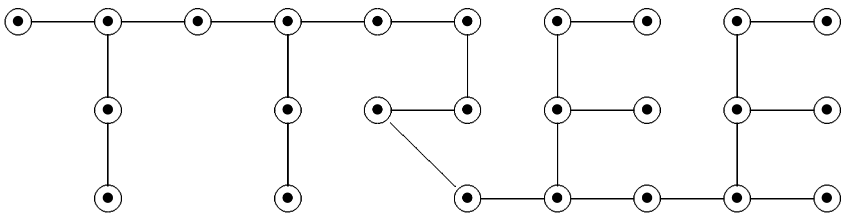
The question of determining whether a hamiltonian cycle exists, on the other hand, turns out to be a rather more difficult problem, and we shall not study this here.

17.5. Trees

For obvious reasons, we make the following definition.

DEFINITION. A graph $T = (V, E)$ is called a tree if it satisfies the following conditions:
(T1) T is connected.
(T2) T does not contain a cycle.

EXAMPLE 17.5.1. The graph represented by the picture



is a tree.

The following three simple properties are immediate from our definition.

PROPOSITION 17D. *Suppose that $T = (V, E)$ is a tree with at least two vertices. Then for every pair of distinct vertices $x, y \in V$, there is a unique path in T from x to y.*

Proof. Since T is connected, there is a path from x to y. Let this be

$$v_0(=x), v_1, \ldots, v_r(=y). \tag{1}$$

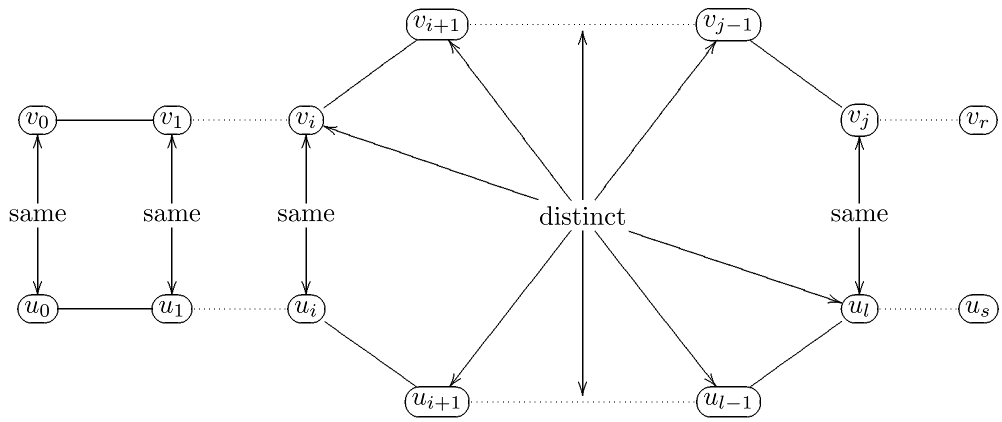
Suppose on the contrary that there is a different path

$$u_0(=x), u_1, \ldots, u_s(=y). \tag{2}$$

We shall show that T must then have a cycle. Since the two paths (1) and (2) are different, there exists $i \in \mathbb{N}$ such that

$$v_0 = u_0, \qquad v_1 = u_1, \qquad \ldots, \qquad v_i = u_i \qquad \text{but} \qquad v_{i+1} \neq u_{i+1}.$$

Consider now the vertices $v_{i+1}, v_{i+2}, \ldots, v_r$. Since both paths (1) and (2) end at y, they must meet again, so that there exists a smallest $j \in \{i+1, i+2, \ldots, r\}$ such that $v_j = u_l$ for some $l \in \{i+1, i+2, \ldots, s\}$. Then the two paths



give rise to a cycle

$$v_i, v_{i+1}, \ldots, v_{j-1}, u_l, u_{l-1}, \ldots, u_{i+1}, v_i,$$

contradicting the hypothesis that T is a tree. ○

PROPOSITION 17E. *Suppose that $T = (V, E)$ is a tree with at least two vertices. Then the graph obtained from T be removing an edge has two components, each of which is a tree.*

Sketch of Proof. Suppose that $\{u, v\} \in E$, where $T = (V, E)$. Let us remove this edge, and consider the graph $G = (V, E')$, where $E' = E \setminus \{\{u, v\}\}$. Define a relation $\mathcal{R}$ on V as follows. Two vertices $x, y \in V$ satisfy $x\mathcal{R}y$ if and only if $x = y$ or the (unique) path in T from x to y does not contain the edge $\{u, v\}$. It can be shown that $\mathcal{R}$ is an equivalence relation on V, with two equivalence classes $[u]$ and $[v]$. We can then show that $[u]$ and $[v]$ are the two components of G. Also, since T has no cycles, so neither do these two components. ○

PROPOSITION 17F. *Suppose that $T = (V, E)$ is a tree. Then $|E| = |V| - 1$.*

Proof. We shall prove this result by induction on the number of vertices of $T = (V, E)$. Clearly the result is true if $|V| = 1$. Suppose now that the result is true if $|V| \leq k$. Let $T = (V, E)$ with $|V| = k + 1$. If we remove one edge from T, then by Proposition 17E, the resulting graph is made up of two components, each of which is a tree. Denote these two components by

$$T_1 = (V_1, E_1) \qquad \text{and} \qquad T_2 = (V_2, E_2).$$

Then clearly $|V_1| \le k$ and $|V_2| \le k$. It follows from the induction hypothesis that

$$|E_1| = |V_1| - 1 \qquad \text{and} \qquad |E_2| = |V_2| - 1.$$

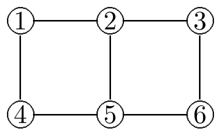
Note, however, that $|V| = |V_1| + |V_2|$ and $|E| = |E_1| + |E_2| + 1$. The result follows. ○

17.6. Spanning Tree of a Connected Graph

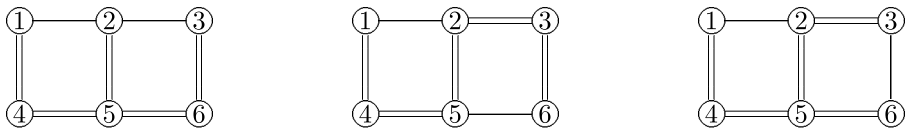
Definition. Suppose that $G = (V, E)$ is a connected graph. Then a subset T of E is called a spanning tree of G if T satisfies the following two conditions:

(ST1) Every vertex in V belongs to an edge in T.
(ST2) The edges in T form a tree.

Example 17.6.1. Consider the connected graph described by the following picture.



Then each of the following pictures describes a spanning tree.



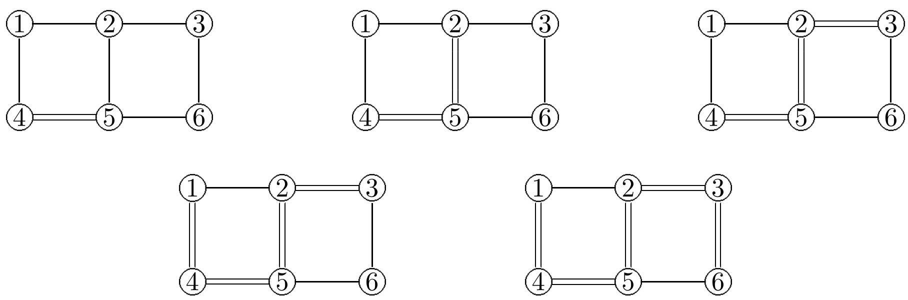
Here we use the notation that an edge represented by a double line is an edge in T. It is clear from this example that a spanning tree may not be unique.

The natural question is, given a connected graph, how we may "grow" a spanning tree. To do this, we apply a "greedy algorithm" as follows.

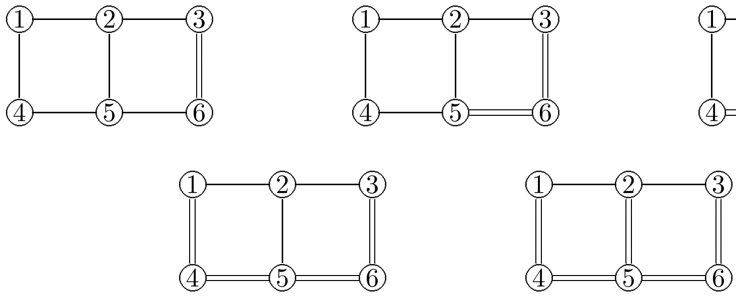
GREEDY ALGORITHM FOR A SPANNING TREE. *Suppose that $G = (V, E)$ is a connected graph.*

(1) Take any vertex in V as an initial partial tree.
(2) Choose edges in E one at a time so that each new edge joins a new vertex in V to the partial tree.
(3) Stop when all the vertices in V are in the partial tree.

Example 17.6.2. Consider the connected graph in Example 17.6.1. Let us start with vertex 5. Choosing the edges $\{4, 5\}, \{2, 5\}, \{2, 3\}, \{1, 4\}, \{3, 6\}$ successively, we obtain the following partial trees, the last of which represents a spanning tree.



However, if we start with vertex 6 and choose the edges $\{3,6\},\{5,6\},\{4,5\},\{1,4\},\{2,5\}$ successively, we obtain the following partial trees, the last of which represents a different spanning tree.



PROPOSITION 17G. *The Greedy algorithm for a spanning tree always works.*

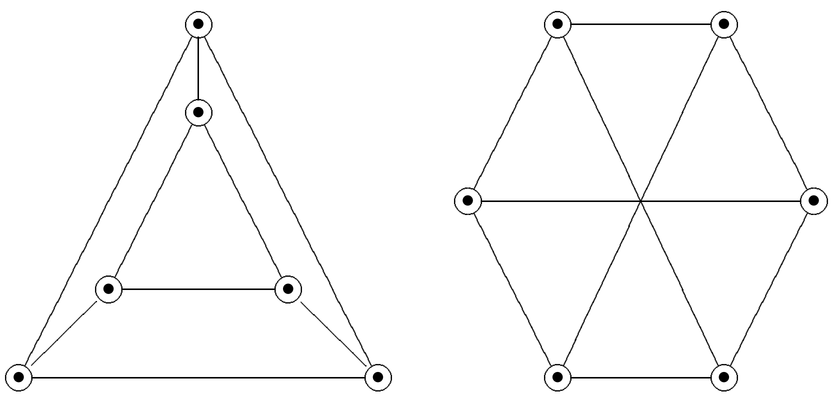
Proof. We need to show that at any stage, we can always join a new vertex in V to the partial tree by an edge in E. To see this, let S denote the set of vertices in the partial tree at any stage. We may assume that $S \neq \emptyset$, for we can always choose an initial vertex. Suppose that $S \neq V$. Suppose on the contrary that we cannot join an extra vertex in V to the partial tree. Then there is no edge in E having one vertex in S and the other vertex in $V \setminus S$. It follows that there is no path from any vertex in S to any vertex in $V \setminus S$, so that G is not connected, a contradiction. ○

Problems for Chapter 17

1. A 3-cycle in a graph is a set of three mutually adjacent vertices. Construct a graph with 5 vertices and 6 edges and no 3-cycles.

2. How many edges does the complete graph K_n have?

3. By looking for 3-cycles, show that the two graphs below are not isomorphic.



4. For each of the following lists, decide whether it is possible that the list represents the valencies of all the vertices of a graph. Is so, draw such a graph.
   a) $2,2,2,3$ b) $2,2,4,4,4$ c) $1,2,2,3,4$ d) $1,2,3,4$

5. Suppose that the graph G has at least 2 vertices. Show that G must have two vertices with the same valency.

6. Consider the graph represented by the following adjacency list.

0	1	2	3	4	5	6	7	8	9
5	2	1	7	2	0	1	3	0	0
8	6	4		6	8	2		5	5
9		6			9	4			

How many components does this graph have?

7. Mr and Mrs Graf gave a party attended by four other couples. Some pairs of people shook hands when they met, but naturally no couple shook hands with each other. At the end of the party, Mr Graf asked the other nine people how many hand shakes they had made, and received nine different answers. Since the maximum number of handshakes any person could make was eight, the nine answers were $0, 1, 2, 3, 4, 5, 6, 7, 8$. Let the vertices of a graph be $0, 1, 2, 3, 4, 5, 6, 7, 8, g$, where g represents Mr Graf and the nine numbers represent the other nine people, with person i having made i handshakes.
   a) Find the adjacency list of this graph.
   b) How many handshakes did Mrs Graf make?
   c) How many components does this graph have?

8. Hamilton and Euler decided to have another holiday. This time they visited a country with 9 cities and a road system represented by the following adjacency table.

1	2	3	4	5	6	7	8	9
2	1	2	1	4	1	2	1	2
4	3	4	3	6	5	6	3	4
6	7	8	5		7	8	7	6
8	9		9		9		9	8

   a) Was either disappointed?
   b) The government of this country was concerned that either of these mathematicians could be disappointed with their visit, and decided to build an extra road linking two of the cities just in case. Was this necessary? If so, advise them how to proceed.

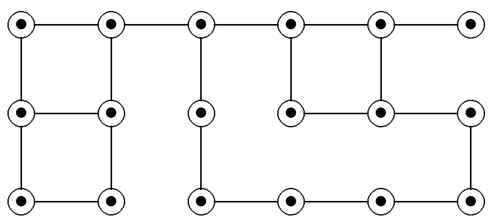
9. Find a hamiltonian cycle in the graph formed by the vertices and edges of an ordinary cube.

10. For which values of $n \in \mathbb{N}$ is it true that the complete graph K_n has an eulerian walk?

11. Draw the six non-isomorphic trees with 6 vertices.

12. Suppose that $T = (V, E)$ is a tree with $|V| \geq 2$. Use Propositions 17A and 17F to show that T has at least two vertices with valency 1.

13. Use the Greedy algorithm for a spanning tree on the following connected graph.



14. Show that there are 125 different spanning trees of the complete graph K_5.