For this reason it usually is important to conduct a thorough **sensitivity analysis** after finding a solution that is optimal under the assumed parameter values. The general purpose is to identify the relatively *sensitive* parameters (i.e., those that cannot be changed much without changing the optimal solution), to try to estimate these more closely, and then to select a solution that remains a good one over the ranges of likely values of the sensitive parameters. This is what the O.R. Department will do for the Wyndor Glass Co. problem, as you will see in Sec. 6.7. However, it is necessary to acquire some more background before finishing that story.

Occasionally, the degree of uncertainty in the parameters is too great to be amenable to sensitivity analysis. In this case, it is necessary to treat the parameters explicitly as *random variables.* Formulations of this kind have been developed, but they are beyond the scope of this book.

As you work through the examples in the next section, you will find it good practice to analyze how well each of the preceding assumptions applies to these problems.

3.4 Additional Examples

The Wyndor Glass Co. problem is a prototype example of linear programming in several respects: It involves allocating limited resources among competing activities, its model fits our standard form, and its context is the traditional one of improved business planning. However, the applicability of linear programming is much wider. In this section we begin broadening our horizons. As you study the following examples, note that it is their underlying mathematical model rather than their context that characterizes them as linear programming problems. Then give some thought to how the same mathematical model could arise in many other contexts by merely changing the names of the activities and so forth.

These examples have been kept very small (by linear programming standards) for ease of reading. However, much larger versions of the problems, involving hundreds of constraints and variables, are readily solvable by linear programming.

REGIONAL PLANNING

One of the interesting social experiments in the Mediterranean region is the system of kibbutzim, or communal farming communities, in Israel. It is common for groups of kibbutzim to join together to share common technical services and to coordinate their production. Our first example concerns one such group of three kibbutzim, which we call the *Southern Confederation of Kibbutzim.*

Overall planning for the Southern Confederation of Kibbutzim is done in its Coordinating Technical Office. This office currently is planning agricultural production for the coming year.

Table 3.3 **Resources data for Southern Confederation of Kibbutzim**

Kibbutz	*Usable land (acres)*	*Water allocation (acre feet)*
1	400	600
2	600	800
3	300	375

Table 3.4 **Crop data for Southern Confederation of Kibbutzim**

Crop	*Maximum quota (acres)*	*Water consumption (acre feet/acre)*	*Net return (dollars/acre)*
Sugar beets	600	3	400
Cotton	500	2	300
Sorghum	325	1	100

The agricultural output of each kibbutz is limited by both the amount of available irrigable land and by the quantity of water allocated for irrigation by the Water Commissioner (a national government official). These data are given in Table 3.3.

The crops suited for this region include sugar beets, cotton, and sorghum, and these are the three being considered for the upcoming season. These crops differ primarily in their expected net return per acre and their consumption of water. In addition, the Ministry of Agriculture has set a maximum quota for the total acreage that can be devoted to each of these crops by the Southern Confederation of Kibbutzim, as shown in Table 3.4.

The three kibbutzim belonging to the Southern Confederation have agreed that every kibbutz will plant the same proportion of its available irrigable land. However, any combination of the crops may be grown at any of the kibbutzim. The job facing the Coordinating Technical Office is to plan how many acres to devote to each crop at the respective kibbutzim while satisfying the given restrictions. The objective is to maximize the total net return to the Southern Confederation as a whole.

FORMULATION AS A LINEAR PROGRAMMING PROBLEM The quantities to be decided upon are the number of acres to devote to each of the three crops at each of the three kibbutzim. The decision variables, x_j $(j = 1,2,\dots,9)$, represent these nine quantities, as shown in Table 3.5. Since the measure of effectiveness Z is total net return, the resulting linear programming model for this problem is

Table 3.5 Decision variables for Southern Confederation of Kibbutzim problem

	Allocation (acres)		
	Kibbutz		
Crop	1	2	3
Sugar beets	x_1	x_2	x_3
Cotton	x_4	x_5	x_6
Sorghum	x_7	x_8	x_9

Maximize $Z = 400(x_1 + x_2 + x_3) + 300(x_4 + x_5 + x_6) + 100(x_7 + x_8 + x_9)$,

subject to the following constraints:

1. *Land*:

$$x_1 + x_4 + x_7 \leq 400$$
$$x_2 + x_5 + x_8 \leq 600$$
$$x_3 + x_6 + x_9 \leq 300$$

2. *Water*:

$$3x_1 + 2x_4 + x_7 \leq 600$$
$$3x_2 + 2x_5 + x_8 \leq 800$$
$$3x_3 + 2x_6 + x_9 \leq 375$$

3. *Crop*:

$$x_1 + x_2 + x_3 \leq 600$$
$$x_4 + x_5 + x_6 \leq 500$$
$$x_7 + x_8 + x_9 \leq 325$$

4. *Social*:

$$\frac{x_1 + x_4 + x_7}{400} = \frac{x_2 + x_5 + x_8}{600}$$

$$\frac{x_2 + x_5 + x_8}{600} = \frac{x_3 + x_6 + x_9}{300}$$

$$\frac{x_3 + x_6 + x_9}{300} = \frac{x_1 + x_4 + x_7}{400}$$

Table 3.6 Optimal solution for Southern Confederation of Kibbutzim problem

	Best allocation (acres)		
	Kibbutz		
Crop	1	2	3
Sugar beets	$133\frac{1}{3}$	100	25
Cotton	100	250	150
Sorghum	0	0	0

Table 3.5 Decision variables for Southern Confederation of Kibbutzim problem

	Allocation (acres)		
	Kibbutz		
Crop	1	2	3
Sugar beets	x_1	x_2	x_3
Cotton	x_4	x_5	x_6
Sorghum	x_7	x_8	x_9

Maximize $Z = 400(x_1 + x_2 + x_3) + 300(x_4 + x_5 + x_6) + 100(x_7 + x_8 + x_9)$,

subject to the following constraints:

1. *Land:*

$$x_1 + x_4 + x_7 \leq 400$$
$$x_2 + x_5 + x_8 \leq 600$$
$$x_3 + x_6 + x_9 \leq 300$$

2. *Water:*

$$3x_1 + 2x_4 + x_7 \leq 600$$
$$3x_2 + 2x_5 + x_8 \leq 800$$
$$3x_3 + 2x_6 + x_9 \leq 375$$

3. *Crop:*

$$x_1 + x_2 + x_3 \leq 600$$
$$x_4 + x_5 + x_6 \leq 500$$
$$x_7 + x_8 + x_9 \leq 325$$

4. *Social:*

$$\frac{x_1 + x_4 + x_7}{400} = \frac{x_2 + x_5 + x_8}{600}$$

$$\frac{x_2 + x_5 + x_8}{600} = \frac{x_3 + x_6 + x_9}{300}$$

$$\frac{x_3 + x_6 + x_9}{300} = \frac{x_1 + x_4 + x_7}{400}$$

Table 3.6 Optimal solution for Southern Confederation of Kibbutzim problem

	Best allocation (acres)		
	Kibbutz		
Crop	1	2	3
Sugar beets	$133\frac{1}{3}$	100	25
Cotton	100	250	150
Sorghum	0	0	0

5. *Nonnegativity*:

$$x_j \geq 0, \quad \text{for } j = 1, 2, \ldots, 9.$$

This completes the model, except that the social constraints are not yet in an appropriate form for a linear programming model because some of the variables are on the right-hand side. Hence their final form[1] is

4. *Social*:

$$3(x_1 + x_4 + x_7) - 2(x_2 + x_5 + x_8) = 0$$
$$x_2 + x_5 + x_8 - 2(x_3 + x_6 + x_9) = 0$$
$$4(x_3 + x_6 + x_9) - 3(x_1 + x_4 + x_7) = 0.$$

The Coordinating Technical Office formulated this model and then applied the simplex method (developed in the next chapter) to find the best solution. The solution they obtained is

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = (133\tfrac{1}{3}, 100, 25, 100, 250, 150, 0, 0, 0),$$

as shown in Table 3.6.

This example illustrates, among other things, how equality constraints can arise naturally in linear programming problems. One feature of the next example is the inclusion of two other nonstandard forms in the model, namely (1) minimizing the objective function and (2) functional constraints with $\geq$ inequalities.

CONTROLLING AIR POLLUTION

The *Nori & Leets Co.*, one of the major producers of steel in its part of the world, is located in the city of Steeltown and is the only large employer there. Steeltown has grown and prospered along with the company, which now employs nearly 50,000 residents. Therefore, the attitude of the townspeople always has been "What's good for Nori & Leets is good for the town." However, this attitude is now changing; uncontrolled air pollution from the company's furnaces is ruining the appearance of the city and endangering the health of its residents.

A recent stockholders' revolt resulted in the election of a new enlightened Board of Directors for the company. These directors are determined to follow socially responsible policies, and they have been discussing with Steeltown city officials and citizens' groups what to do about the air pollution problem. Together they have worked out stringent air quality standards for the Steeltown airshed.

The three main types of pollutants in this airshed are particulate matter, sulfur oxides, and hydrocarbons. The new standards require that the company

[1] Actually, any one of these equations is redundant and can be deleted if desired. Because of these equations, any two of the land constraints also could be deleted.

Table 3.7 **Clean air standards for Nori & Leets Co.**

Pollutant	*Required reduction in annual emission rate (million pounds)*
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

Table 3.8 **Reduction in emission rate from maximum feasible use of abatement method for Nori & Leets Co.**

	Taller smokestacks		*Filters*		*Better fuels*	
Pollutant	*Blast furnaces*	*Open-hearth furnaces*	*Blast furnaces*	*Open-hearth furnaces*	*Blast furnaces*	*Open-hearth furnaces*
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

reduce its annual emission of these pollutants by the amounts shown in Table 3.7. The Board of Directors has instructed management to have the engineering staff determine how to achieve these reductions in the most economical way.

The steel works has two primary sources of pollution, namely, the blast furnaces for making pig iron and the open-hearth furnaces for changing iron into steel. In both cases the engineers have decided that the most effective types of abatement methods are (1) increasing the height of the smokestacks,[1] (2) using filter devices (including gas traps) in the smokestacks, and (3) including cleaner high-grade materials among the fuels for the furnaces. All these methods have technological limits on how much emission they can eliminate, as shown (in millions of pounds per year) in Table 3.8.

However, the methods can be used at any fraction of their abatement capacities shown in this table. Because they operate independently, the emission reductions achieved by each method are not substantially affected by whether or not the other methods also are used.

After these data were developed, it became clear that no single method by

[1] Subsequent to this study, this particular abatement method has become a controversial one. Because its effect is to reduce ground-level pollution by spreading emissions over a greater distance, environmental groups contend that this creates more acid rain by keeping sulfur oxides in the air longer. Consequently, the U.S. Environmental Protection Agency adopted new rules in 1985 to remove incentives for using tall smokestacks.

Table 3.9 **Total annual cost from maximum feasible use of abatement method for Nori & Leets Co.**

Abatement method	*Blast furnaces*	*Open-hearth furnaces*
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

itself could achieve all the required reductions. On the other hand, combining all three methods at full capacity (which would be prohibitively expensive if the company's products are to remain competitively priced) is much more than adequate. Therefore, the engineers concluded that they would have to use some combination of the methods, perhaps with fractional capacities, based upon their relative costs. Furthermore, because of the differences between the blast and the open-hearth furnaces, the two types probably should not use the same combination.

An analysis was conducted to estimate the total annual cost that would be incurred by each abatement method. In addition to increased operating and maintenance expenses, consideration was given also to the initial costs (converted to an equivalent annual basis) of the method as well as any resulting loss in efficiency of the production process. This analysis led to the total cost estimates (in millions of dollars) given in Table 3.9 for using the methods at their full abatement capacities. It also was determined that the cost of a method being used at a lower level is essentially proportional to its fractional capacity. Thus, for any given fraction used, the total annual cost would be that fraction of the corresponding quantity in Table 3.9.

The stage now was set to develop the general framework of the company's plan for pollution abatement. This plan would consist of specifying which types of abatement methods would be used and at what fractions of their abatement capacities for (1) the blast furnaces and (2) the open-hearth furnaces. Because of the combinatorial nature of the problem of finding a plan that satisfies the requirements with the smallest possible cost, an operations research team was formed to solve the problem. The team adopted a linear programming approach, formulating the model summarized next.

FORMULATION AS A LINEAR PROGRAMMING PROBLEM This problem has six decision variables, x_j ($j = 1,2,\ldots,6$), each representing the usage of one of the three abatement methods for one of the two types of furnaces, expressed as a fraction of the abatement capacity. The ordering of these variables is shown in Table 3.10. Because the objective is to minimize total cost while satisfying the emission reduction requirements, the model is

Table 3.10 **Decision variables (fraction of maximum feasible use of abatement method) for Nori & Leets Co.**

Abatement method	*Blast furnaces*	*Open-hearth furnaces*
Taller smokestacks	x_1	x_2
Filters	x_3	x_4
Better fuels	x_5	x_6

$$\text{Minimize} \quad Z = 8x_1 + 10x_2 + 7x_3 + 6x_4 + 11x_5 + 9x_6,$$

subject to the following constraints:

1. *Emission reduction*:

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \geq 60$$
$$35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 \geq 150$$
$$37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 \geq 125$$

2. *Technological*:

$$x_j \leq 1, \quad \text{for } j = 1, 2, \ldots, 6$$

3. *Nonnegativity*:

$$x_j \geq 0, \quad \text{for } j = 1, 2, \ldots, 6.$$

The operations research team used this model[1] to find the minimum cost plan, $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0.623, 0.343, 1, 0.048, 1)$. Sensitivity analysis then was conducted, followed by detailed planning and managerial review. Soon after, this program for controlling air pollution was fully implemented by the company, and the citizens of Steeltown breathed deep sighs of relief.

OTHER EXAMPLES

The three linear programming examples you have seen so far are but a small sampling of the uses of this technique. Many more illustrations are given in Chaps. 7 and 8; most involve business and industrial applications, but several others arise in different contexts. Chap. 7 focuses on certain special types of linear programming problems that provide many important applications. Chap. 8 considers some examples that are more difficult to formulate, and it also includes a case study involving the design of school attendance zones to achieve better racial balance. But before considering these topics, we next discuss how to solve linear programming problems.

[1] An equivalent formulation can express each decision variable in natural units for its abatement method; for example, x_1 and x_2 could represent the number of *feet* that the heights of the smokestacks are increased.