4

Integer Programming

IN THE LINEAR programming problems considered so far, the variables have been permitted to assume all nonnegative real values. However, there are many problems in which the variables must assume only integer values. For example, it would be meaningless to have an answer calling for the manufacture of half a table or for the chartering of 1.2 airplanes. In some problems, such as the transportation problem with integer values for supply and demand, the simplex method will yield integer answers; however, in many other problems it will not. In this chapter we formulate a number of problems that require integer variables and present three algorithms for solving these integer programming problems.

4.1 EXAMPLES

EXAMPLE 1 (THE TRANSPORTATION PROBLEM). Suppose a manufacturer making one product has m factories and n warehouses. The demand at the jth warehouse is d_j, $j = 1, 2, \ldots, n$, and the supply available from the ith factory is s_i, $i = 1, 2, \ldots, m$. The cost of shipping one unit from the ith factory to the jth warehouse is c_{ij}. Our problem is to determine the amount, x_{ij}, of the product to be sent from the ith factory to the jth warehouse.

If we assume that the total supply at least equals the total demand,

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j,$$

so that our problem is feasible, then the mathematical model is

$$\text{Minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq s_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j, \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \quad \text{and} \quad \text{integral},$$

$$i = 1, 2, \ldots, m; \qquad j = 1, 2, \ldots, n.$$

If this problem is converted to standard form, then the only entries in the constraint matrix are 1s, −1s, and 0s. It can be shown using a result of Hoffman and Kruskal that in this case the simplex method will automatically yield integer solutions if the s_i and d_j are integers. However, the simplex method is a rather poor way of solving the transportation problem. In Chapter 5 we present a special algorithm for this problem that is rather efficient. This algorithm was developed because the transportation model arises repeatedly in practice. △

EXAMPLE 2 (THE KNAPSACK PROBLEM). Consider the problem faced by a hiker who cannot carry more than k pounds of equipment. She has n items that she is considering bringing. To each item she assigns a relative value, c_j, with the most important items having the highest values. Let a_j be the weight of the jth item. The hiker's problem is to decide which of the n items to carry; she will choose those that maximize the total relative value subject to the weight limitation.

To construct the mathematical model, let $x_j = 1$ if the jth item is chosen and let $x_j = 0$ if the jth item is not chosen. Then the model is

$$\text{Maximize} \quad z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_j x_j \leq k$$

$$x_j = 0 \quad \text{or} \quad 1, \quad j = 1, 2, \ldots, n.$$

Note that by limiting the value of x_j to 0 or 1, the left-hand side of the constraint represents just the weight of the items that are chosen. This type of an integer programming problem is called a **zero–one programming problem.** △

EXAMPLE 3 (THE ASSIGNMENT PROBLEM). Suppose n people, P_1, $P_2, \ldots, P_n$, are being considered for n jobs, J_1, $J_2, \ldots, J_n$. Using various criteria, including past performance, aptitude, and job ability, we specify a value c_{ij} that would accrue if the ith person is given the jth job. We assume that each person is assigned to exactly one job and that each job is assigned to exactly one person. Our problem is to assign the people to the jobs so that the total value of the assignment is maximized.

To construct the mathematical model, define the variables x_{ij} so that

$$x_{ij} = \begin{cases} 1 & \text{if } P_i \text{ is assigned to } J_j \\ 0 & \text{otherwise.} \end{cases}$$

Then the mathematical model is

$$\text{Maximize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n \tag{2}$$

$$x_{ij} = 0 \quad \text{or} \quad 1, \quad i, j = 1, 2, \ldots, n.$$

Under the condition that x_{ij} has a value of either 0 or 1, exactly one of the summands in Equation (1) can be nonzero, and, likewise, exactly one of the summands in Equation (2) can be nonzero. Constraint (1) says that job

j is assigned to exactly one person; constraint (2) says that person i is assigned to exactly one job. Just as in the transportation problem, the result of Hoffman and Kruskal applies and the simplex algorithm yields a zero–one solution to the assignment problem. However, there is a special algorithm that efficiently handles this problem; it will be discussed in Chapter 5. △

EXAMPLE 4 (THE TRAVELING SALESMAN PROBLEM). A traveling salesman has to visit each of n cities, $C_1, C_2, \ldots, C_n$. He must start from his home office in city C_1 and return to C_1 after visiting each city exactly once. Such a route is called a **tour**. The order in which he visits cities $C_2, C_3, \ldots, C_n$ does not matter. He knows the distance between each pair of cities and wants to choose a tour that minimizes the total distance traveled.

To formulate the mathematical model, let c_{ij} be the distance between C_i and C_j. Let the variable x_{ij} be defined by

$$\begin{aligned} x_{ij} &= 1 \quad \text{if the route includes traveling from } C_i \text{ to } C_j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

The condition that the route must go to exactly one city after leaving C_i may be written

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n.$$

The condition that the route goes through every city exactly once can be phrased by saying that each city must be reached from exactly one city, or

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n.$$

Our mathematical model is then

$$\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n \tag{3}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n \tag{4}$$

$$x_{ij} = 0 \quad \text{or} \quad 1, \quad i, j = 1, 2, \ldots, n.$$

Consider the feasible solution for this problem when $n = 12$:

$$x_{12} = x_{23} = x_{34} = x_{45} = x_{56} = x_{61} = 1$$
$$x_{78} = x_{89} = x_{9,10} = x_{10,11} = x_{11,12} = x_{12,7} = 1$$

and

$$x_{ij} = 0 \quad \text{for all other values of } i \text{ and } j.$$

This solution is feasible, since each index from 1 to 12 occurs exactly once in the first position and exactly once in the second position. However, it is not an acceptable solution, since there are two disconnected subtours. We must design a way to eliminate disconnected routes from our set of feasible solutions.

To this end we introduce $n - 1$ new variables, $u_2, u_3, \ldots, u_n$, and $(n-1)^2 - (n-1)$ new constraints. The constraints are

$$u_i - u_j + nx_{ij} \le n - 1, \quad i, j = 2, 3, \ldots, n, \quad \text{and} \quad i \ne j \tag{5}$$
$$u_i \ge 0 \quad \text{and} \quad \text{integral}, \quad i = 2, 3, \ldots, n.$$

Before we had $2n$ constraints and n^2 variables; these variables had values of either 0 or 1 and n of them x_{ii} were always 0. We now have

$$2n + (n-1)^2 - (n-1) = n^2 - n + 2$$

linear constraints and

$$n^2 + n - 1$$

integer-valued variables.

We now show that the constraints (3), (4), and (5) do not permit disconnected routes and still include all routes satisfying the original problem statement. First we assume that there is a subtour; that is, the route leads back to C_1 before visiting all the cities. Then there must be another subtour, since each city is visited exactly once. This subtour will start and end at some city in the list $C_2, C_3, \ldots, C_n$; it will not include C_1; and it will include $r \le n - 1$ cities. The r variables x_{ij} that describe this subtour will be equal to 1. We add up the r constants (5) that correspond to these nonzero x_{ij}. This new constraint is satisfied by any solution that satisfies (5). As we take the sum to form this new constraint, we have $-u_j$ when the route enters city C_j and $+u_j$ when it leaves. Since the route enters and leaves each of the r cities exactly once, the u_j's cancel out in the sum. Thus, the new constraint is

$$nr \le (n-1)r,$$

which is a contradiction of our assumption that there was a subtour of length $r \le n - 1$.

For example, if we had the subtour starting at C_4,

$$C_4 \to C_5 \to C_3 \to C_2 \to C_4,$$

so that

$$x_{45} = x_{53} = x_{32} = x_{24} = 1,$$

then we would form our new constraint by adding the constraints

$$\begin{aligned} u_4 - u_5 + nx_{45} &\le n - 1 \\ u_5 - u_3 + nx_{53} &\le n - 1 \\ u_3 - u_2 + nx_{32} &\le n - 1 \\ u_2 - u_4 + nx_{24} &\le n - 1 \end{aligned}$$

and obtain

$$4n \le 4(n - 1).$$

We have now shown that constraints (3), (4), and (5) allow no subtours. Now we show that these constraints do not exclude any potential routes. To do this we show that each u_i can be assigned a nonnegative integer value for any route and that these values satisfy the constraints given in (5).

Let t_i be the position in the route at which C_i is visited. Thus, $t_1 = 1$ for C_1. If we consider the route that starts $C_1 \to C_4 \to C_6 \to C_2 \to \cdots$, then $t_1 = 1$, $t_4 = 2$, $t_6 = 3$, $t_2 = 4, \ldots$. Let $u_i = t_i$ for $i = 2, 3, \ldots, n$. We show that for each i and j, (5) holds. Either $x_{ij} = 1$ or $x_{ij} = 0$. If $x_{ij} = 1$, then C_j is visited immediately after C_i, so that

$$t_j = t_i + 1.$$

Substituting this equation into (5), we have

$$u_i - u_j + nx_{ij} = t_i - (t_i + 1) + n = n - 1$$

as we needed. If $x_{ij} = 0$, then since $u_i \le n$ and $u_j \ge 2$, we have

$$u_i - u_j \le n - 2 \le n - 1,$$

so that (5) holds.

We have shown that a model for the traveling salesman problem is

$$\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n$$

$$u_i - u_j + nx_{ij} \le n - 1, \quad i, j = 2, 3, \ldots, n, \quad \text{and} \quad i \ne j$$

$$x_{ij} = 0 \quad \text{or} \quad 1, \quad i, j = 1, 2, \ldots, n$$

$$u_i \ge 0 \quad \text{and} \quad \text{integral}, \quad i = 2, 3, \ldots, n. \qquad \triangle$$

EXAMPLE 5 (STOCK CUTTING PROBLEM). A plumber can buy plastic pipe in 6- and 12-ft lengths. The current job requires eight 4-ft lengths, five 5-ft lengths, and three 7-ft lengths. The plumber wants to figure out how many of each of the two stock lengths should be bought to minimize waste.

We determine all the possible ways the stock lengths can be cut to yield the necessary lengths. A 6-ft piece can be cut to give

one 4-ft length and 2 ft of scrap	(cutting pattern 1)
one 5-ft length and 1 ft of scrap	(cutting pattern 2).

A 12-ft piece can be cut to give

one 4-ft piece and 8 ft of scrap	(cutting pattern 3),
two 4-ft pieces and 4 ft of scrap	(cutting pattern 4),
three 4-ft pieces	(cutting pattern 5),
one 4-ft piece, one 5-ft piece, and 3 ft of scrap	(cutting pattern 6),
one 4-ft piece, one 7-ft piece, and 1 ft of scrap	(cutting pattern 7),
one 5-ft piece and 7 ft of scrap	(cutting pattern 8),
two 5-ft pieces and 2 ft of scrap	(cutting pattern 9),
one 7-ft piece and 5 ft of scrap	(cutting pattern 10),
one 7-ft piece and one 5-ft piece	(cutting pattern 11).

Let piece 1 be of length $l_1 = 4$ ft, let piece 2 be of length $l_2 = 5$ ft, and let piece 3 be of length $l_3 = 7$ ft. Let

a_{ij} = number of pieces of length l_i in cutting pattern j;
b_i = number of pieces of length l_i which are needed;
c_j = waste in cutting pattern j;
x_j = number of times cutting pattern j is used.

Our mathematical model is

$$\begin{gathered}\text{Minimize}\quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x}\\ \text{subject to}\\ \mathbf{A}\mathbf{x} = \mathbf{b}\\ \mathbf{x} \geq \mathbf{0} \quad \text{and} \quad \text{integral,}\end{gathered}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{c}^{\mathrm{T}} = [2 \quad 1 \quad 8 \quad 4 \quad 0 \quad 3 \quad 1 \quad 7 \quad 2 \quad 5 \quad 0]$$

and

$$\mathbf{b} = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}. \qquad \triangle$$

EXAMPLE 6 (FIXED CHARGE PROBLEM). A manufacturing corporation makes n products, and naturally the board of directors wants to minimize manufacturing costs. Each unit of product j that is made costs c_j dollars to produce (raw materials, labor, direct machine costs, etc.). Moreover, if any units of product j are made, there is a fixed cost of k_j dollars, which represents the initial cost of setting up the production and distribution process.

Let x_j be the number of units of product j that are made. Suppose that the production process is constrained by a system of inequalities such as that in Exercises 1 or 2 in Section 1.1. Our objective function is

$$\text{Minimize} \quad z = \sum_{j=1}^{n} (c_j x_j + k_j y_j)$$

where the production constraints not involving the new variables y_i hold and in addition

$$y_j = \begin{cases} 1 & \text{if} \quad x_j > 0 \\ 0 & \text{if} \quad x_j = 0. \end{cases} \tag{6}$$

That is, y_j indicates whether any of the jth product is manufactured. The constraints in (6) are nonlinear functions of x_j and y_j. These constraints are not defined by hyperplanes as they must be for a linear programming problem.

However, this problem can be cast as a problem with *linear* constraints in which some of the variables are restricted to be integers and others may take any value. Such problems are called **mixed integer programming problems**.

Suppose we know an upper bound on the number of units of x_i that can be produced. That is, suppose we know numbers M_j, such that

$$x_j \le M_j, \quad j = 1, 2, \ldots, n.$$

We now show that we may reformulate the definition of y_j in (6) as

$$\left.\begin{aligned} y_j &\geq \frac{x_j}{M_j} \\ y_j &= 0 \quad \text{or} \quad 1. \end{aligned}\right\} \tag{7}$$

If $x_j > 0$, then $y_j \geq x_j/M_j > 0$ implies that $y_j = 1$ since y_j can be only 0 or 1. If $x_j = 0$, then $y_j \geq x_j/M_j \geq 0$, so that $y_j = 0$ or 1. But since we are minimizing, the objective function will be smaller if $y_j = 0$. Therefore, at the minimum value of the objective function, if $x_j = 0$, then $y_j = 0$. The constraints given by (7) are now linear. We combine (7) with those constraints describing the production process to obtain our mixed integer programming model. △

EXAMPLE 7 (EITHER–OR PROBLEM). Suppose that we have a situation in which either the constraint

$$\sum_{j=1}^{n} a_{1j}x_j \leq b_1 \tag{8}$$

or the constraint

$$\sum_{j=1}^{n} a_{2j}x_j \leq b_2 \tag{9}$$

holds. We can convert this condition to one in which the constraints are linear if we have available numbers M_1 and M_2, such that

$$\sum_{j=1}^{n} a_{1j}x_j - b_1 \leq M_1$$

and

$$\sum_{j=1}^{n} a_{2j}x_j - b_2 \leq M_2$$

for all feasible solutions.

Let y be a zero–one variable. Consider the problem

$$\sum_{j=1}^{n} a_{1j}x_j - b_1 \leq M_1 y \tag{10}$$

$$\sum_{j=1}^{n} a_{2j}x_j - b_2 \leq M_2(1 - y) \tag{11}$$

$$y = 0 \quad \text{or} \quad 1.$$

If $y = 0$ in our new problem constraint, then (10) is the same as constraint (8), and constraint (11) is redundant, since it holds for all feasible solutions. If $y = 1$, then constraint (11) is the same as constraint (9), and constraint (10) is redundant. △

We now examine a general integer programming problem and describe some methods for solving it. We consider the following problem:

$$\begin{aligned} &\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{subject to} \\ &\qquad \mathbf{Ax} \le \mathbf{b} \\ &\qquad \mathbf{x} \ge \mathbf{0} \\ &x_j = \text{integer} \quad \text{if} \quad j \in I, \end{aligned}$$

where I is a subset of $\{1, 2, \dots, n\}$. If $I = \{1, 2, \dots, n\}$, then the problem is called a **pure integer programming problem**. If I is a proper subset of $\{1, 2, \dots, n\}$, then the problem is called a **mixed integer programming problem**. In a pure integer programming problem every variable is required to be an integer. In a mixed integer programming problem only some of the variables are required to have integer values. Examples 6 and 7 are mixed integer programming problems. Examples 1–5 are pure integer programming problems. In Examples 2 and 3 the variables are restricted to the values 0 or 1.

One might attempt to solve an integer programming problem by treating it as a linear programming problem (that is, by not restricting the variables to integer values) and then rounding the answer to the nearest integer. Under extremely fortunate circumstances one might not have to round at all. But there are other situations in which rounding will produce an incorrect answer.

EXAMPLE 8. Consider the integer programming problem

$$\begin{aligned} &\text{Maximize} \quad z = 7x + 8y \\ &\text{subject to} \\ &\qquad 10x + 3y \le 52 \\ &\qquad 2x + 3y \le 18 \\ &x \ge 0, \quad y \ge 0, \quad \text{and integers.} \end{aligned}$$

If we ignore the restriction that x and y are integers, the simplex method gives the solution (verify)

$$x = 4\tfrac{1}{4}, \quad y = 3\tfrac{1}{6}$$

with optimal value

$$z = 55\tfrac{1}{12}.$$

If we round the values of x and y to the nearest integer values that are feasible, we get

$$x = 4, \quad y = 3,$$

and

$$z = 52.$$

However, the solution

$$x = 3, \quad y = 4$$

is also feasible, and the value of the objective function for this solution is

$$z = 53. \qquad \triangle$$

4.1 EXERCISES

In Exercises 1–6 formulate the given problem as an integer programming problem.

1. **Equipment purchasing problem.** A ribbon manufacturer is considering the purchase of two different types of printing machines that will be used to emboss designs on the ribbon. Machine A can print 100 m per minute and requires 50 m^2 of floor space, whereas machine B can print 200 m per minute and requires 140 m^2 of floor space. Suppose that the manufacturer must print at least 600 m per minute and has no more than 350 m^2 of floor space. If a model A machine costs $22,000 and a model B machine costs $48,000, how many machines of each type should be bought to minimize the cost?
2. **A production problem.** A chair manufacturer makes three different types of chairs, each of which must go through sanding, staining, and varnishing. In addition, the model with the vinyl-covered back and seat must go through an upholstering process. The following table gives the time required for each operation on each type of chair, the available time for each operation in hours per month, and the profit per chair for each model. How many chairs of each type should be made to maximize the total profit?

Model	*Sanding (hr)*	*Staining (hr)*	*Varnishing (hr)*	*Upholstering (hr)*	*Profit ($)*
A—solid back and seat	1.0	0.5	0.7	0	10
B—ladder back, solid seat	1.2	0.5	0.7	0	13
C—vinyl-covered back and seat	0.7	0.3	0.3	0.7	8
Total time available per month	600	300	300	140	

3. Pam Hardy currently has six favorite country and western songs. There are 10 compact disks that contain different groups of these songs available. Suppose that the jth CD costs c_j dollars. Set up a model that Pam could use to

determine the cheapest selection of CDs to buy to get at least one version of each of her favorite songs.

4. Tommy Jones's mother is planning his 10th birthday party and will serve a variety of soft drinks, which will be chosen from the list below.

Drink	*Cola*	*Root beer*	*Cherry*	*Lemon*	*Orange*	*Grape*	*Ginger ale*
Price per bottle (cents)	69	59	62	62	65	55	65

From past experience it has been determined that at least 12 bottles of soft drinks are needed. Also, at least 2 bottles of ginger ale, at least 2 bottles of cola, and no more than 3 bottles of fruit-flavored soft drinks are needed. How many bottles of each type should be bought to minimize the total cost?

5. A manager for a large corporation must prepare a list of projects that her group will complete over the next year. She has under consideration 10 such projects but will not be able to do all of them because of limits on personnel and budget. She has assigned a weight to each project that represents to her the value of completing the project. The personnel, capital requirements, and weights for each project are given in the following table.

	Project									
	1	2	3	4	5	6	7	8	9	10
Person-weeks	250	195	200	70	30	40	100	170	40	120
Cost (thousands of dollars)	400	300	350	100	70	70	250	250	100	200
Value of completion	70	50	60	20	10	20	30	45	10	40

The manager has available 1000 person-weeks and $1,500,000 to allocate among the projects. Which projects should she choose to complete to maximize the value?

6. Each day at the Graphic Arts Co. the press operator is given a list of jobs to be done during the day. He must determine the order in which he does the jobs based on the amount of time it takes to change from one job setup to the next. Clearly he will arrange the jobs in an order that minimizes the total setup time. Assume that each day he starts the press from a rest state and returns it to that state at the end of the day. Suppose on a particular day that he must do six jobs for which he estimates the changeover times given in the following table. What schedule of jobs should the operator use?

	j						
i	1	2	3	4	5	6	*Rest*
	⟨From job i to job j (min)⟩						
1	0	10	5	15	10	20	5
2	10	0	10	10	20	15	10
3	5	5	0	5	10	10	15
4	8	10	3	0	9	14	10
5	4	7	8	6	0	10	10
6	10	5	10	15	10	0	8
Rest	7	7	9	12	10	8	0

4.1 PROJECTS

1. Meg Watkins is trying to decide which college to attend. From among the applications that she submitted, she has been admitted to four schools. One is a large state university that is about 250 miles from her home. At this school she may live in a dormitory for two years, but then must find accommodations in the community. Another is a small private school about 1000 miles from her home that has an excellent reputation. At this school there are dormitory accommodations for all students. Meg, under pressure from her father, also applied to and was accepted by the private church-related school in her hometown. Since she is a local student, the school would expect her to live at home.

   Another possibility open to Meg is to go to the local community college for two years and then transfer to another school. The reputation of the community college has been improving over the last few years. The state university would carry forward her acceptance for two years and transfer all credits. The distant private school will also carry forward her acceptance but most likely will transfer nine credits fewer than two full years of credit. The local private school has no formal statement on transfer from two-year schools. The accompanying table gives the cost for attending each school and Meg's assessment of the worth (or utility) of having certain blocks of credit from each school.

	State	*Distant private*	*Local private*	*Community college*
Tuition	\$2500/year	\$14,000/year	\$9000/year	—
Living				
On campus	\$3500/year	\$3500/year	\$125/month	\$125/month
Off campus	\$4000/year			
Humanities	7	9	6	6
Social science	6	8	5	6
Science	8	5	4	3
Major	8	10	6	—

Set up a model that Meg could use to maximize the future worth of her education assuming that she can earn \$3000/summer and that her father will provide \$6000/year for her education and living expenses. You may wish to consider allowing Meg the option of working while going to school or of going to summer school for certain courses.

2. Consider the problem of making change in a supermarket. Suppose that the cash register shows that your change is to be C cents. Initially, we assume $C < 100$. The coins that are available to make change have values

$$w_1 = 1, \quad w_2 = 5, \quad w_3 = 10, \quad w_4 = 25, \quad \text{and} \quad w_5 = 50.$$

   (a) Set up an integer programming problem for finding which combination of coins yields the correct change using the smallest number of coins.
   (b) Construct a table giving the solution to part (a) for $C = 1, 2, \ldots, 99$.
   (c) One algorithm for change-making calls for giving as many of the largest-denomination coins as possible, then using the next largest denomination for the remaining amount, and so on. Does this algorithm give the correct answer for each $C = 1, 2, \ldots, 99$?
   (d) Suppose our monetary system had coins with values

$$w_1 = 1, \quad w_2 = 5, \quad w_3 = 20, \quad \text{and} \quad w_4 = 25.$$

   Use the algorithm in part (c) to make up $C = 40$. Is this a minimal solution?
   (e) Consider the problem of giving C dollars in change, where $C = 1, 2, \ldots, 9$. Would it be advantageous to use a \$2 bill in addition to the \$1 and \$5 bills?
   (f) What other situations have models similar to the change-making problem?

Further Reading

Chang, S. K., and Gill, A. "Algorithmic Solution of the Change-Making Problem." *J. ACM* **17** (1970), 113–122.

Hoffman, A. J., and Kruskal, J. B. "Integral Boundary Points of Convex Polyhedra," in *Linear Inequalities and Related Systems* (H. W. Kuhn and A. W. Tucker, Eds.), pp. 223–246. Princeton Univ. Press, Princeton, NJ, 1956.

4.2 CUTTING PLANE METHODS

In this section we discuss one approach that has been used to solve integer programming problems. The algorithms for such problems are not as nice as those for linear programming problems in the sense that there is not one algorithm that works well for all integer programming problems. Among the difficulties with these algorithms is their inefficiency for even medium-sized problems. For example, computations for the traveling salesman problem (Example 4 in Section 4.1) become prohibitively long for over 200 cities.

Consider the pure integer programming problem

$$\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{1}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{2}$$

$$\mathbf{x} \geq \mathbf{0} \quad \text{and} \quad \text{integral}, \tag{3}$$