Existence and stability of equilibria in OLG models under adaptive expectations*

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Abstract

In this paper we deal with an Overlapping Generations Model with production under three diverse assumptions about agents rationality; rational, adaptive and myopic expectations. We determine a uniqueness condition for stationary steady states in the model with perfect foresight which rests on the second derivatives of the production and utility functions. Such condition results to be more restrictive than the one developed for the model with myopic expectations which, due to the correspondence among steady states of the three models, could be considered as an alternative. Further, we completely develop the analysis of the model under adaptive expectations. We derive stability conditions and determine the bifurcation diagram in all the three cases. From the comparison it results that stability conditions for the case with rational expectations are less restrictive than for both adaptive and myopic ones. We notice that, differently from what happens in the OLG model of pure exchange, the adaptive expectations do not improve local stability performances of the model with respect to myopic expectations; this is due to the fact that in our two-dimensional model a Neimark-Hopf bifurcation could arise, cutting off part of the parameter space which results to be stable in the myopic case.

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1 Introduction

The Overlapping Generations Model, first introduced by Allais [1] in 1947, became after the 1958 article by Paul Samuelson [9] *"...the most important and influential paradigm in neoclassical general equilibrium theory outside of the Arrow-Debreu economy"* (quoting Geanakoplos [7]). As suggested by the name, in this model we suppose that at any time different generations of individuals are living; each generation being characterised by a representative agent. Every generation may trade with other ones which are in different periods of their life. Samuelson's innovation was in postulating a demographic structure in which generations overlap indefinitely into the future. The capital stock is generated by individuals who save during their working lives to finance their consumption when retired. This structure makes it possible to study the aggregate implications of life-cycle saving by individuals, the determinants of the aggregate capital stock, the effects of government policy about national debt, social security, taxation as well as the effect of bequests on the accumulation of capital. A vast literature about the Phillips curve, the business cycle and the foundations of monetary theory is also based on the model.

Over the year the OLG model has been developed into a general equilibrium model with many agents, multiple kinds of commodities and production, completely founded on the neoclassical methodological assumptions of agent optimization, market clearing and rational expectations as the Arrow-Debreu model. In this more comprehensive version of Samuelson's original idea, known as the *Overlapping Generations Model of General Equilibrium*, equilibria show profound differences with respect to that of the Arrow-Debreu model: the OLG provides an example of an economy in which the competitive equilibrium may be not efficient or Pareto optimal (Samuelson itself pointed out this fact in his 1958 paper [9]), money may have positive value, there are robust economies with a continuum of equilibria (indeterminacy of equilibrium in the one-commodity case was first studied by Gale [5]) and finally the core of an OLG economy may be empty. None of this could happen in any Arrow-Debreu economy

In this paper we consider an Overlapping Generations Model with production, first studied by Diamond in his famous 1965 paper [4], in the version given by Galor and Ryder in 1989 [6]. In Section 2 the set-up of the model is fully described. In Section 3 we specify three diverse expectations functions. In Subsection 3.1 we consider fully rational agents and discuss all the results about existence, uniqueness and stability of non-trivial steady states obtained by Galor and Ryder [6]. In Subsection 3.2 we present the work of Michel and de la Croix [8] who studied the stability properties of stationary steady states under the hypothesis of myopic expectations. Both in Subsection 3.1 and in Subsection 3.2 some minor improvements and results are discussed and in Subsection 3.3 is proposed an alternative uniqueness condition for the model with perfect foresight. In Subsection 3.4 we develop and analyze the model under the assumption of adaptive expectations. We study the relationships among steady states of this and of the two alternative models. We also develop a complete local stability analysis. In Section 3.5 a comparative discussion of stability conditions of the three models is performed. Finally in the Appendix we have left two, less interesting, proofs.

2 The model

We refer to the classical Diamond's OLG model with production [4] as developed and studied by Galor and Ryder in [6]. We consider an infinite horizon and perfectly *competitive* economy where a single good is produced by means of capital and labor. We suppose that the economy exists in discrete time and that individuals live for two periods. For a given rate of population growth $n \geq -1$, the endowment of labor at time t is exogenously set by

$$L_t = (1+n)^t L_0 \tag{1}$$

Moreover, at each time t the endowment of capital is increased by the quantity of resources not consumed in the previous period and (i.e. $Y_{t-1} - C_{t-1}$), for a given rate of capital depreciation $0 \leq \delta \leq 1$, is

$$K_t = Y_{t-1} + (1-\delta) K_{t-1} - C_{t-1} \tag{2}$$

where C_{t-1} is the aggregate consumption and Y_{t-1} is the aggregate production, both at time $t-1$. Output will satisfy

$$Y_t = F(K_t, L_t) \tag{3}$$

where the unchanging technology is supposed to perform constant returns to scale, so that we can write

$$y_t = f(k_t) \tag{4}$$

where $k_t = \frac{K_t}{L_t}$ and $y_t = \frac{Y_t}{L_t}$ correspond to the pro-capite amount of capital and output, respectively and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the intensive form production function. As in [6], we suppose that such a neoclassical production function is twice continuously differentiable, positive, increasing and strictly concave

$$f(k) > 0, \qquad f'(k) > 0, \qquad f''(k) < 0, \qquad \text{for } k > 0 \tag{5}$$

The competitive assumption in both the capital and the labor market implies that in equilibrium the interest rate, r_t, and the wage, w_t, are respectively equal to the marginal product of capital and labor. Formally,

$$r_t = f'(k_t) \tag{6}$$
$$w_t = f(k_t) - k_t f'(k_t) \triangleq w(k_t). \tag{7}$$

In the first period of their life, agents work and allocate the resulting income, w_t, between consumption, c_t^1, and savings, s_t, that is

$$w_t = c_t^1 + s_t \tag{8}$$

Such savings earn the return r in the following period and are spent in the older age when agents are retired. The olds consume both their capital income and their existing wealth, hence we assume *non-altruistic* agents (no bequest is considered). The consumption in the second period is

$$c_{t+1}^2 = (1 + r_{t+1} - \delta) s_t, \tag{9}$$

which together with (8) gives the budget constraint

$$c_t^1 + \frac{1}{1 + r_{t+1} - \delta} c_{t+1}^2 = w_t. \tag{10}$$

Individual preferences are characterized by their intertemporal utility function $u\left(c_t^1, c_{t+1}^2\right)$ defined over their, non-negative, consumption in the two periods of their life. We assume that

A1) $u : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous on $\mathbb{R}_+^2$ and twice continuous differentiable on $\mathbb{R}_{++}^2$.[1]

Moreover, we require that such utility function satisfies:

A2) the *no satiation* property, which means that it is increasing in both variables,

$$u_1\left(c^1, c^2\right) > 0 \quad \text{and} \quad u_2\left(c^1, c^2\right) > 0, \quad \forall c^1, c^2 > 0, \tag{11}$$

where the subscripts indicate, respectively, the partial derivative with respect to the first and second argument;

A3) a decreasing marginal rate of substitution, or alternatively, the requirement of *strictly quasi concavity* on the interior of the consumption set.

[1] We define $\mathbb{R}_+^2 \triangleq [0, +\infty) \times [0, +\infty)$ and $\mathbb{R}_{++}^2 \triangleq (0, +\infty) \times (0, +\infty)$.

Moreover, the Hessian of u, $H_u(c^1, c^2)$, is *non-singular* for each $c^1, c^2 > 0$[2], that is

$$\det H_u\left(c^1, c^2\right) \neq 0, \quad \forall c^1, c^2 > 0, \tag{12}$$

where

$$H_u\left(c^1, c^2\right) \triangleq \begin{pmatrix} u_{11}\left(c^1, c^2\right) & u_{12}\left(c^1, c^2\right) \\ u_{21}\left(c^1, c^2\right) & u_{22}\left(c^1, c^2\right) \end{pmatrix}. \tag{13}$$

A4) the condition

$$u_1 u_{12} > u_2 u_{11}, \quad \forall c^1, c^2 > 0, \tag{14}$$

which guarantees that consumption in period two is a *normal good* (see Lemma 1);

A5) and finally, *avoided starvation* in both periods, which is guaranteed by

$$\begin{aligned} \lim_{c^1 \downarrow 0} u_1\left(c^1, c^2\right) &= \infty, \quad \forall c^2 > 0, \\ \lim_{c^2 \downarrow 0} u_2\left(c^1, c^2\right) &= \infty, \quad \forall c^1 > 0. \end{aligned} \tag{15}$$

Given all these assumptions agents solve their consumption-saving choice problem and obtain

$$s_t = s\left(w_t, r_{t+1}^e\right) = \arg\max_{s_t \in [0, w_t]} u\left[w_t - s_t, \left(1 + r_{t+1}^e - \delta\right) s_t\right] \tag{16}$$

where r_{t+1}^e is the $t+1$ expected rate of return, that is not known at the time in which the decision has to be taken.

Lemma 1 *Let assumptions A1)-A5) hold, then for each $w > 0$ and $r > 0$*

$$s_w(w, r) \triangleq \frac{\partial}{\partial w} s(w, r) > 0.^3 \tag{17}$$

Proof. See Appendix. ■

Under this set of assumption the economic dynamic is characterized by the following set of equations

$$M \triangleq \begin{cases} k_{t+1} = \frac{s\left(w_t, r_{t+1}^e\right)}{1+n} \\ w_t = f(k_t) - k_t f'(k_t) \triangleq w(k_t) \\ r_{t+1} = f'(k_{t+1}) \end{cases} \tag{18}$$

[2] This is exclusively a technical assumption which will be used later on in order to apply the implicit function theorem. We recall that even strict concavity would not be sufficient to imply $\det H_u\left(c^1, c^2\right) \neq 0$; for example, the strictly concave function $F(x, y) = -x^4 - y^4$ is such that $\det H_F(0, 0) = 0$.

[3] We have omitted all the indexes for notation convenience.

which gives

$$k_{t+1} = \frac{s\left[f\left(k_t\right) - k_t f'\left(k_t\right), f'\left(k^e_{t+1}\right)\right]}{1+n}. \tag{19}$$

In the next section we study this model under three different expectation formation mechanisms.

3 Different expectations formation mechanisms

Now we need to close the model through an adequate assumption about the expectations formation mechanism. In what follows we study the model under three alternative specifications.

3.1 Rational expectations

In [6], Galor and Ryder study the model of Section 2 assuming that the agents are fully rational. In this case they are able to predict the future evolution of the economic variables so that their expectations are independent of past observations and will perform *perfect foresight*, that is

$$r^e_{t+1} = r_{t+1} \tag{20}$$

As established in [6], such *self-fulfilling* expectations will *not* be, in general, *uniquely* determined unless some further conditions are satisfied.

The system M together with the perfect foresight assumption (20), gives

$$k_{t+1} = \frac{s\left[f\left(k_t\right) - k_t f'\left(k_t\right), f'\left(k_{t+1}\right)\right]}{1+n}. \tag{21}$$

A6) We assume that Inada conditions are satisfied at the origin

$$\lim_{k\downarrow 0} f\left(k\right) = 0 = f(0) \qquad \text{and} \qquad \lim_{k\downarrow 0} f'\left(k\right) = \infty. \tag{22}$$

It is possible to show that (see [6] for details)

Proposition 1 (Galor and Ryder (1989)) *Given $k_t > 0$, and provided that savings are a non-decreasing function of the interest rate, that is*

$$\frac{\partial s\left(w_t, r^e_{t+1}\right)}{\partial r^e_{t+1}} \geq 0, \tag{23}$$

then there exists a unique self-fulfilling expectations $k^e_{t+1} = k_{t+1} > 0$.

Lemma 2 *Let assumptions A1), A2), A3), and A5) hold, then, for each* $w > 0$ *and* $r > 0$, *condition (23) is satisfied if and only if*

$$u_1 u_2 \geq (u_2 u_{12} - u_1 u_{22})\, s_t \,(1 + r_{t+1} - \delta)^{4}. \tag{24}$$

Proof. See Appendix. ■

Therefore, under condition (23), there exists a single valued function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $k_{t+1} = \phi(k_t)$ and $\phi(k_t) \downarrow 0$ as $k_t \downarrow 0$. To see these latter properties it suffices to note that

$$0 \leq \phi(k_t) = k_{t+1} = \frac{s\left(w_t, r^e_{t+1}\right)}{1+n} \leq \frac{w(k_t)}{1+n} = \frac{f(k_t) - k_t f'(k_t)}{1+n} \leq \frac{f(k_t)}{1+n} \tag{25}$$

and use the assumption A6).

Now, in order to show further results, let us state some preliminary definitions.

Definition 1 *A sequence* $\{k_t\}$ *is said a* dynamic equilibrium *if*

$$k_{t+1} = \frac{s\left[f(k_t) - k_t f'(k_t), f'(k_{t+1})\right]}{1+n} \tag{26}$$

with k_0 *exogenously given.*

Definition 2 *A stationary capital-labor ratio,* $\bar{k}$ *is said a* steady state equilibrium *if*

$$\bar{k} = \frac{s\left[f\left(\bar{k}\right) - \bar{k} f'\left(\bar{k}\right), f'\left(\bar{k}\right)\right]}{1+n} \tag{27}$$

Suppose now that there is an upper bound to the attainable capital, $\tilde{k}$, for our technology

$$f\left(\tilde{k}\right) = (1+n)\,\tilde{k} \tag{28}$$

This fact, together with the other assumptions about production and utility functions, grants us that every trajectory of the system will eventually remains boxed in a given, bounded, range; in particular we have the three following cases

$$\begin{array}{ll} \text{if } k_t \geq \tilde{k} & \text{then } 0 < k_{t+1} < k_t \\ \text{if } 0 < k_t < \tilde{k} & \text{then } 0 < k_{t+1} < \tilde{k} \\ \text{if } k_t = 0 & \text{then } k_{t+1} = 0_t \end{array} \tag{29}$$

[4] From an economic point of view, condition (24) is equivalent to require that the substitution effect created by an increase in the interest rate is not smaller (in absolute value) than the income effect.

which implies that a steady state equilibrium exists (at least the trivial one $\bar{k} = 0$) and that all steady state equilibria lie in the interval $\left[0, \tilde{k}\right)$.

All the assumption we have introduced are not sufficient to rule out the possibility that the only steady state equilibrium in this overlapping generations economy is the trivial one, characterised by zero production and consumption. In order to avoid this unsatisfactory occurrence it's necessary to strengthen the Inada conditions. We have:

Proposition 2 (Galor and Ryder (1989)) *Given the previously described overlapping generations model, if*

$$\lim_{k \to 0} \left[-kf''(k)\right] > 1 + n \tag{30}$$

then there is $\hat{k} > 0$ *such that*

$$\lim_{t \to \infty} k_t \geq \hat{k} \qquad \textit{for all } k_0 > 0 \tag{31}$$

So, under the assumption (30), which is stronger than the Inada condition $\lim_{k \to 0} f'(k) = \infty$, the trivial steady state cannot occur provided that the initial amount of capital is strictly positive. Observe that, even with such strenghtened Inada condition, the existence of a non-trivial steady state is not guaranteed. As Galor and Ryder pointed out, in order to obtain such a result, some restrictions on the nature of the interactions between preferences and technology are needed. The following propositions can be proved (see [6] for details):

Proposition 3 (Galor and Ryder (1989)) *The overlapping-generation economy previously described experiences non-trivial steady state equilibrium if* $k_0 > 0$ *and*

a) $\lim_{k \to 0} \frac{-s_w k f''(k)}{1+n-s_r f''(k)} > 1$,

b) $\lim_{k \to \infty} f'(k) = 0$,

c) $s_r(w, r) \geq 0$, *for all* $(w, r) \geq 0$.

Proposition 4 (Galor and Ryder (1989)) *Given* $k_0 > 0$ *and the following conditions:*

a) $\lim_{k \to 0} \frac{-s_w k f''(k)}{1+n-s_r f''(k)} > 1$,

b) $\lim_{k \to \infty} f'(k) = 0$,

c) $s_r(w, r) \geq 0$, *for all* $(w, r) \geq 0$,

d) $\phi'(k) \geq 0$, *for all* $k > 0$,

e) $\phi''(k) \leq 0$, *for all* $k > 0$,

the overlapping generations economy has a unique, non-trivial, and globally stable steady state equilibrium.

Remark 1 *We observe that under assumption A4) (that is the assumption that future consumption is a normal good) Assumption d) in Proposition 4 is redundant. Moreover, despite the correctness of Proposition 4, its proof is not really correct.*

So, in a standard overlapping generations model with capital accumulation, there is in general a multiplicity of stationary equilibria.

3.2 Myopic expectations

Recently in [8] Michel and de la Croix have studied the model of Section 2 by assuming *myopic* foresight, that is

$$r_{t+1}^{e} = r_t = f'(k_t). \tag{32}$$

In this case the inter-temporal equilibrium with initial capital stock $k_0 > 0$ is *necessarily* unique and its dynamics is characterized by the following first order difference equation

$$k_{t+1} = \frac{s\left[f(k_t) - k_t f'(k_t), f'(k_t)\right]}{1+n} \triangleq m(k_t). \tag{33}$$

The derivative of $m(k)$ is

$$m'(k) = \frac{s_w\left[w(k), f'(k)\right] w'(k) + s_r\left[w(k), f'(k)\right] f''(k)}{1+n} \tag{34}$$

As a matter of fact, in [8] the authors compare their model with the one studied in [6]. They first prove the following

Proposition 5 (Michel and de la Croix (2000)) *The two dynamics with perfect and myopic foresight have the same positive steady states.*

Remark 2 *We observe that under the assumptions made in [6] (which are the ones we will adopt throughout the paper), in particular, the hypothesis that $f(0) = 0$ (cf. 22), $k = 0$ is also a corner steady state*[5] *for both the dynamics. Indeed, whenever the production function is absolutely continuous and such that $f(0) = 0$ then we always have*

$$w(0+) \triangleq \lim_{k \downarrow 0} f(k) - k f'(k) = 0. \tag{35}$$

[5] For a dynamics described by $k_{t+1} = g(k_t)$, with $g = m, \phi$, we say that $k = 0$ is a *corner steady state* if

$$g(0+) \triangleq \lim_{x \downarrow 0} g(x) = 0.$$

To see this notice that the absolute continuity of f implies

$$f(k) - f(0) = \int_0^k f'(v)dv < +\infty, \tag{36}$$

which, in turn, yields

$$\lim_{k \downarrow 0} kf'(k) = 0. \tag{37}$$

Therefore, the fact that $f(k_t) - k_t f'(k_t) = w_t \to 0$ as $k_t \to 0$ together with the relation $s\left(w_t, r_{t+1}^e\right) \leq w_t$, true, by construction, for both the dynamics, yield

$$0 \leq k_{t+1} = \frac{s\left(w_t, r_{t+1}^e\right)}{1+n} \leq \frac{w_t}{1+n}, \tag{38}$$

hence $k = 0$ is a corner steady state in both cases.

Remark 3 *Michel and de la Croix show, by a counter-example, that, in a slightly more general model than the one in [6], when $k = 0$ is a corner steady state of the dynamics with myopic foresight it is not necessarily a corner steady state of the dynamics with perfect foresight. The example they provide make use of a CES production function with $0 < \rho < 1$[6]. Clearly this production function is such that $f(0) > 0$.*

Michel and de la Croix compare the local stability of both dynamics and prove the following

Proposition 6 (Michel and de la Croix (2000)) *Let $\bar{k} > 0$ a steady state and assume (23). If $m'(\bar{k}) \geq 0$ (monotonic dynamics with myopic foresight), $\bar{k}$ is* stable, unstable *or* non-hyperbolic *for the dynamics with perfect foresight if and only if it is respectively* stable, unstable *or* non-hyperbolic *for the dynamics with myopic foresight.*

Remark 4 *Notice that when $m'(\bar{k}) < 0$ the equivalence stated in the previous proposition in not generally true but we can only say that if $-1 < m'(\bar{k}) < 0$ then $\left|\phi'(\bar{k})\right| < 1$. In [8] the authors do not list all the assumptions on f and*

[6]The *constant elasticity of substitution* or CES production function has the form

$$Y = F(K, L) = (aK^\rho + bL^\rho)^{\frac{1}{\rho}},$$

with a, b positive constants and $\rho \neq 0$ (see [11], p. 19). Michel and de la Croix simply require $\rho > 0$, but with $\rho \geq 1$ the example given in [8] does not work. It does work if $0 < \rho < 1$.

u except for what they call ***Assumption H***[7] *and it seems they say that the dynamics with perfect foresight are strictly monotonic under this assumption, that is* $\phi'(\bar{k}) > 0$ *for each steady state* $\bar{k} > 0$*. We observe that this is not generally true unless further conditions are assumed; in particular, we need conditions which ensure* $s_w(w,r) \geq 0$ *for each* $(w,r) \in \mathbb{R}_+^2$*, for instance assumption A4).*

Finally, in [8] the authors prove the following uniqueness result:

Proposition 7 (Michel and de la Croix (2000)) *No more than one positive steady state* $\bar{k}$ *of the dynamics with perfect foresight exists if*

$$s_w\left[w(k), f'(k)\right] w'(k) + s_r\left[w(k), f'(k)\right] f''(k) < \frac{s\left[w(k), f'(k)\right]}{k}, \quad \forall k > 0. \tag{39}$$

Such a steady state exists if and only if

$$\lim_{k \downarrow 0} \frac{s\left[w(k), f'(k)\right]}{k} > 1 + n. \tag{40}$$

Remark 5 *Comparing Proposition 7 with Proposition 4 we point out that Michel and de la Croix require conditions only on the second derivatives of the functions u and f but, contrary to Galor and Ryder, they can prove only uniqueness. As a matter of fact, if Michel and de la Croix wanted global stability together with uniqueness they should set conditions on the third derivatives just as Galor and Ryder do (we are talking about the assumpion* $\phi''(k) \leq 0$*).*

3.3 Steady state uniqueness under perfect foresight: A note

In this subsection we provide, without referring to the myopic analysis, a sufficient uniqueness condition in the perfect foresight case which rests only on the second derivatives of the functions u and f . In Subsection 3.1 we saw that, under suitable assumptions (see Proposition 1), equation (21) implicitly

[7] In [8] Assumption H is stated as follows: For all $(k,w) \in \mathbb{R}_+^2$,

$$g(k,w) = 0 \Longrightarrow g_k(k,w) > 0,$$

where

$$g(k,w) \triangleq (1+n)k - s(w, f'(k)).$$

defines a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $k_{t+1} = \phi(k_t)$. That is, ϕ is such that, for each $k_t > 0$,

$$\phi(k_t) = \frac{s\left[w(k_t), f'(\phi(k_t))\right]}{1+n}. \tag{41}$$

It is possible to prove the following

Proposition 8 *Let condition (23) and assumptions A1), A2), A3) and A5) hold, then the dynamics under perfect foresight admits at most one positive steady state if*

$$\frac{s_w(w(x), f'(y))\, w'(x)}{1+n-s_r(w(x), f'(y))\, f''(y)} < \frac{y}{x}, \quad \forall x, y > 0 \ s.t. \ y \leq w(x). \tag{42}$$

Proof. $\bar{k} > 0$ is a steady state with perfect foresight if $\phi\left(\bar{k}\right) = \bar{k}$, that is if

$$\frac{s\left[w(\bar{k}), f'\left(\phi\left(\bar{k}\right)\right)\right]}{1+n} = \bar{k} \tag{43}$$

or, equivalently, if $\bar{k}$ is such that

$$\frac{s\left[w(\bar{k}), f'\left(\phi\left(\bar{k}\right)\right)\right]}{\bar{k}} = 1+n. \tag{44}$$

Clearly, a sufficient condition for uniqueness of $\bar{k}$ is $h'(k) < 0$, $\forall k > 0$, where the function h is thus defined

$$h(k) \triangleq \frac{s\left[w(k), f'(\phi(k))\right]}{k} = \frac{(1+n)\,\phi(k)}{k}. \tag{45}$$

An easy calculation yields

$$h'(k) = \left(\frac{1+n}{k}\right)\left(\phi'(k) - \frac{\phi(k)}{k}\right). \tag{46}$$

Since,

$$\phi'(k) = \frac{s_w\left[w(k), f'(\phi(k))\right] w'(k)}{1+n-s_r\left[w(k), f'(\phi(k))\right] f''(\phi(k))}, \tag{47}$$

it follows that $h'(k) < 0$, $\forall k > 0$, if and only if

$$\frac{s_w\left[w(k), f'(\phi(k))\right] w'(k)}{1+n-s_r\left[w(k), f'(\phi(k))\right] f''(\phi(k))} < \frac{\phi(k)}{k}; \tag{48}$$

hence the conclusion. ■

Remark 6 *Uniqueness condition given in Proposition 7 is weaker than the one given in Proposition 8. This is due to the fact that if we do not refer to the myopic analysis then we need to consider all the possible implicit function* ϕ *defined by (21), although we know that* $0 < \phi(k) < w(k)$, $\forall k > 0$, *whereas in the myopic case* ϕ *is explicitly given by* m *(cf. (33)).*

Remark 7 *We further observe that a sufficient condition for having* h *strictly decreasing,* $\forall k > 0$, *is, not surprisingly (cf. Proposition 4), the strict concavity of* ϕ *(together with its differentiability). Indeed, if* ϕ *is differentiable strictly concave*[8] *then, for each* $k, k_0 \geq 0$, *we have*

$$\phi(k_0) - \phi(k) \leq \phi'(k)(k_0 - k). \tag{49}$$

For $k_0 = 0$, *we obtain*

$$\phi'(k)k - \phi(k) \leq 0, \tag{50}$$

for each $k > 0$ *(recall that from (25) we have* $\phi(0) = 0$*), that is* $h'(k) \leq 0$, $\forall k > 0$. *Now, observe that* h *cannot be constant in any interval* $I \subset (0, +\infty)$. *In fact, if this were the case then* $h'(k) = 0$, $\forall k \in I$, *and* ϕ *would be a straight line in that interval, which is a contraddiction since* ϕ *is strictly concave. Hence,* ϕ *is strictly decreasing.*

We now turn to the adaptive expectation model.

3.4 Adaptive expectations

Let's consider the well known *first order autoregressive adaptive expectations* learning mechanism. In this case forecasts are formed through a weighted mean, with geometrically decreasing weights, of past observed data. Indeed, in an economy with infinite past, given the time series of old interest rates $\mathbf{r} = (r_t, r_{t-1}, \ldots)$ and set $0 < \alpha < 1$, we can write

$$r^e_{t+1} = \sum_{j=0}^{\infty} \alpha (1-\alpha)^j r_{t-j} = r^e_t + \alpha (r_t - r^e_t) \tag{51}$$

Introducing this *expectation function* in the overlapping generations model we obtain

$$k_{t+1} = \frac{s\left(w_t, r^e_{t+1}\right)}{1+n} \tag{52a}$$

$$w_t = f(k_t) - k_t f'(k_t) \triangleq w(k_t) \tag{52b}$$

$$r_t = f'(k_t) \tag{52c}$$

$$r^e_{t+1} = r^e_t + \alpha (r_t - r^e_t) \tag{52d}$$

[8] We, actually apply the definition of concavity to the function ϕ extended by continuity at the origin by setting $\phi(0) = 0$.

where $f(k_t)$ is the production function defined in (4). Substituting (52b) in (52a) and (52c) in (52d) the system reduces to

$$AM \triangleq \begin{cases} k_{t+1} = \frac{s\left[w(k_t), r^e_{t+1}\right]}{1+n} \\ r^e_{t+1} = r^e_t + \alpha\left[f'(k_t) - r^e_t\right] \end{cases} \tag{53}$$

The following proposition holds.

Proposition 9 *A point* $\bar{k} > 0$ *is a* stationary steady state *for the model under* ***perfect foresight*** *if and only if* $\left(\bar{k}, f'\left(\bar{k}\right)\right)$ *is a* stationary steady state *for the model under* ***adaptive expectations****.*

Proof. In fact, given

$$\bar{k} = \frac{s\left[f\left(\bar{k}\right) - \bar{k}f'\left(\bar{k}\right), f'\left(\bar{k}\right)\right]}{1+n} \tag{54}$$

and set $\bar{r} = f'\left(\bar{k}\right)$, we have

$$\begin{cases} \bar{k} = \frac{s\left[f\left(\bar{k}\right) - \bar{k}f'\left(\bar{k}\right), \bar{r}\right]}{1+n} \\ \bar{r} = \bar{r} + \alpha\left[f'\left(\bar{k}\right) - \bar{r}\right] \end{cases} \tag{55}$$

Likewise, the opposite follows immediatly. ■

Remark 8 *It is useful to notice that, in order to be a stationary steady state for the dynamic with adaptive expectations, a point* $\left(\bar{k}, \bar{r}\right)$ *should satisfy* $\bar{r} = f'\left(\bar{k}\right)$*; so there is a one to one correspondence between steady states of the two maps.*

Remark 9 *Notice that the condition (54) is the same for the model with myopic expectations so that such one to one correspondence among non trivial steady states holds also between the myopic and the adaptive model.*

As a consequence of Proposition 9 and Remark 9, existence and uniqueness of steady states for the model with adaptive expectations is guaranteed under the same assumptions of propositions 3 and 7. Now, suppose that such a (non trivial) steady state exists. We want to compare the local stability properties of the model, subject to the three diverse hypothesis about the expectations formation mechanism.

The Jacobian matrix of AM, evaluated in the steady state, is given by

$$J_{AM} \triangleq \begin{pmatrix} \frac{s_w w' + s_r \alpha f''}{1+n} & \frac{s_r(1-\alpha)}{1+n} \\ \alpha f'' & (1-\alpha) \end{pmatrix} \tag{56}$$

and defining ξ and μ as

$$\begin{cases} \xi \triangleq \frac{s_w w'}{1+n} \\ \mu \triangleq \frac{s_r f''}{1+n} \end{cases} \tag{57}$$

we obtain

$$\begin{cases} \text{trace}\,(J_{AM}) = \xi + \alpha\mu + 1 - \alpha \\ \det\,(J_{AM}) = (1-\alpha)\,\xi \end{cases} \tag{58}$$

Remark 10 *Notice that the assumptions made so far imply $\xi > 0$ and $\mu \leq 0$. Hence, in the sequel the analysis will deal only with these parameter values.*

Given the usual stability conditions for 2×2 discrete dynamical system

$$\begin{cases} \det A < 1 \\ 1 - trA + \det A > 0 \\ 1 + trA + \det A > 0 \end{cases} \tag{59}$$

it results

$$\begin{cases} \xi < \frac{1}{(1-\alpha)} \\ \mu < 1 - \xi \\ \mu > -\frac{(2-\alpha)(1+\xi)}{\alpha} \end{cases} \tag{60}$$

where, transforming the inequalities into equations, the three conditions represent the Neimark-Hopf, Saddle-node and Period-doubling bifurcation loci, respectively. Furthermore, the condition

$$\begin{gathered} (\text{trace}\,(J_{AM}))^2 - 4\det\,(J_{AM}) < 0 \\ \Updownarrow \\ (\xi + \alpha\mu + 1 - \alpha)^2 - 4\,(1-\alpha)\,\xi < 0 \end{gathered} \tag{61}$$

determines the region of parameters for which the system has complex eigenvalues. The situation is summarized in the following Proposition

Proposition 10 *Suppose that $(\bar{k}, \bar{r})$ is a stationary steady state for the OLG economy under adaptive expectations described in (53) then:*

a. For all $\alpha \in (0,1)$, the point $(\bar{k}, \bar{r})$ is locally stable for the system AM iff the pair (ξ, μ) satisfies conditions (60).

b. For all $\alpha \in (0,1)$, the point $(\bar{k}, \bar{r})$ loose stability in one of the following ways:

i. through a Period-doubling bifurcation which occurs when $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$;

ii. through a Saddle-node bifurcation occurring for $\mu = 1 - \xi$;

iii. through a Neimark-Hopf bifurcation by crossing the line $\xi = \frac{1}{(1-\alpha)}$.

A pictorial representation of the bifurcation diagram for the system AM is given in figure 1. The lines colored in magenta, blue and green represent the Period-doubling, Saddle-node and Neimark-Hopf bifurcation loci, respectively; the parabola (in red) marks the limits of the region with complex eigenvalues. Gray is the region of stability, both with real (light) and with complex (dark) eigenvalues.

It is of some interest to investigate how the stability region evolves with the changing in the learning parameter α. The Period-doubling bifurcation locus, given by $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$, converges to the line $\xi = -1$ when $\alpha \to 0$ while converges to the line $\mu = \xi - 1$ when $\alpha \to 1$ (see figure 2). At the same time, the Neimark-Hopf bifurcation locus, given by $\xi = \frac{1}{(1-\alpha)}$, converges to the line $\xi = 1$ when $\alpha \to 0$ while goes away to infinity when $\alpha \to 1$ (see figure 3). Opposite to the previous two cases, the Saddle-node bifurcation locus remains unchanged for all α.

As regards the region with complex eigenvalues, we observe that it is bounded by the conic of equation $(\xi + \alpha\mu + 1 - \alpha)^2 - 4(1-\alpha)\xi = 0$. At a first glance we can observe that it is a matter of parabola due to the fact that the matrix of the quadratic form

$$\begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix} \tag{62}$$

is singular for all α. Such a parabola converges to the lines $\xi = 1$ and $\mu = -\xi$ when $\alpha \to 0$ and $\alpha \to 1$ respectively; evolving between this two limiting situations it first enlarges and subsequently squeezes one against the other its two branches while continuously rotating its symmetry axis counterclock-wise (see figure 4). We analize more precisely its shape in the following paragraph, while studying the linear transformation we are using in the parameter space.

Now we are ready to understand how the stability region changes with respect to the values of α. In figure 5 is shown the evolution of the region and is put in evidence the intersection point between the Neimark-Hopf and Period-Doubling bifurcation lines. The importance of such points will become clear with the following Lemma

Lemma 3 *Given the system AM, for all $\alpha \in (0,1)$, the segment which joins the Origin to the intersection point of the lines $\xi = \frac{1}{(1-\alpha)}$ and $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ belongs to the stability region (except for the extrema in the intersecton point).*

Proof. It is an immediate consequence of the inequalitites in (60) and of the following four facts.

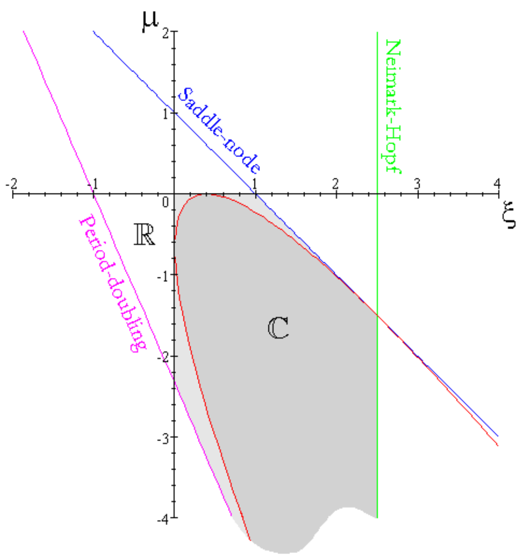

Figure 1: The bifurcation diagram for the system AM with $\alpha = 0.6$

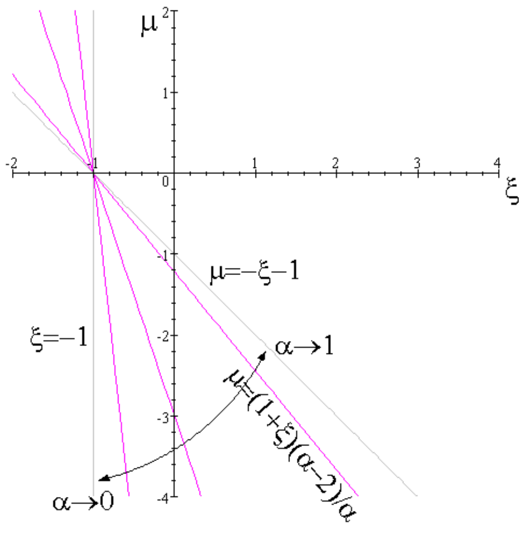

Figure 2: The Period-doubling bifurcation locus drawn for $\alpha = 0.2$, 0.5 and 0.9

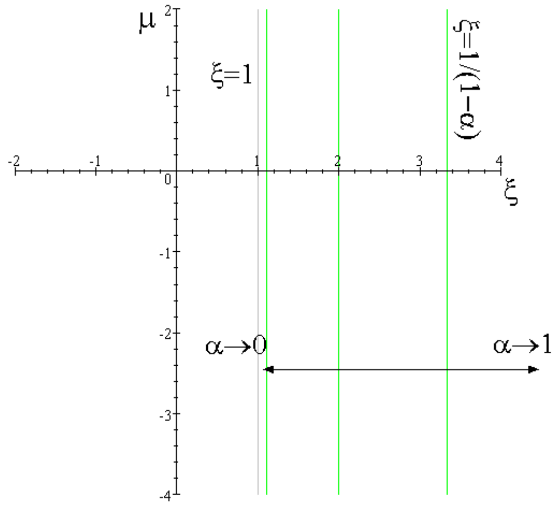

Figure 3: The Neimark-Hopf bifurcation locus drawn for α = 0.1, 0.5 and 0.7

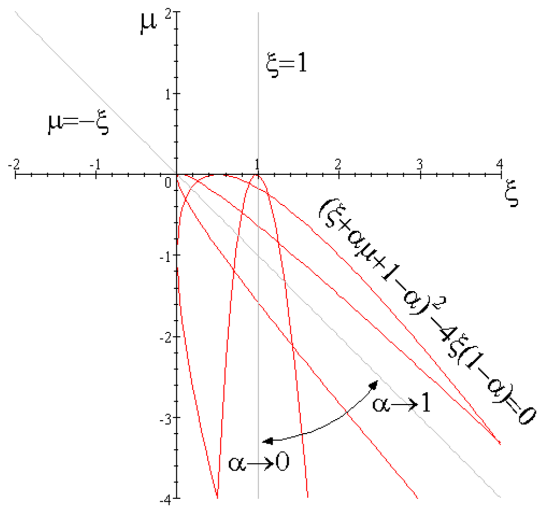

Figure 4: The border of the region with complex eigenvalues drawn for α = 0.02, 0.5 and 0.95

1. The intersection point of the line $\mu = -\xi + 1$ with the $y - axis$ is bigger than 0 (Of course it is $y = 1$!)

2. The intersection point of the line $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ with the $y - axis$ is smaller than 0 (Of course it is $y = \frac{(\alpha-2)}{\alpha}$!)

3. The slope of the line $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ is smaller than the slope of the line $\mu = -\xi + 1$ (Indeed $\frac{(\alpha-2)}{\alpha} < -1$ for all $\alpha \in (0,1)$)

4. Finally, $\frac{1}{(1-\alpha)} > 0$ for all $\alpha \in (0,1)$.

Facts 1. 2. and 4. together with conditions (60) are sufficient to guarantee that the Origin belongs to the stability region; Facts 1. 2. 3. and 4. together with conditions (60) are sufficient to guarantee that the intersection point of the Neimark-Hopf and Period-Doubling bifurcation lines is on the boundary of the stability region. So, as the stability region is convex (it is the intersection of convex sets), we have the result. (See figure (6) for a pictorial representation). ■

Lemma 4 *Given the system AM, for all $\alpha \in (0,1)$, the segment which joins the point $(1,0)$ to the intersection point of the lines $\xi = \frac{1}{(1-\alpha)}$ and $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ belongs to the stability region (except for the extrema).*

Proof. It is an immediate consequence of the inequalitites in (60) and of the following four facts.

1. The point $(1,0)$ is the intersection between the Saddle-Node bifurcation line ($\mu = -\xi + 1$) and the $x - axis$

2. The intersection point of the line $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ with the $y - axis$ is smaller than 0 (Of course it is $y = \frac{(\alpha-2)}{\alpha}$!)

3. The slope of the line $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ is smaller than the slope of the line $\mu = -\xi + 1$ (Indeed $\frac{(\alpha-2)}{\alpha} < -1$ for all $\alpha \in (0,1)$)

4. Finally, $\frac{1}{(1-\alpha)} > 1$ for all $\alpha \in (0,1)$.

Indeed such conditions are sufficient to guarantee that both the point $(1,0)$ and the intersection point of the Neimark-Hopf and Period-Doubling bifurcation lines are on the boundary of the stability region. Furthermore, as the stability region is convex (it is the intersection of convex sets), the segment belongs to its convex closure. The possibility that the segment be

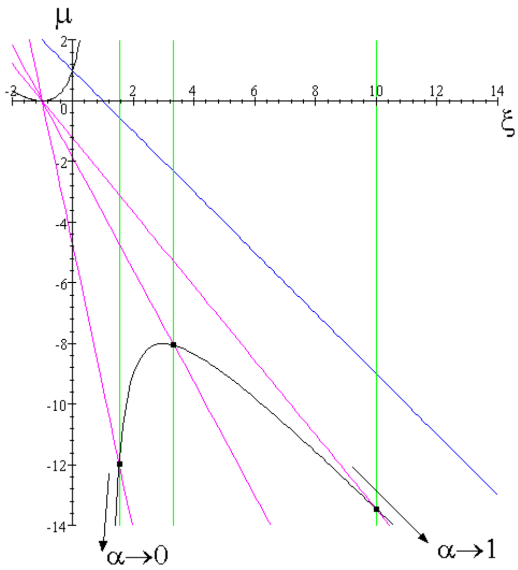

Figure 5: The intersection points between the Neimark-Hopf and Period-Doubling bifurcation lines for $\alpha = 0.35,\ 0.7$ and 0.9.

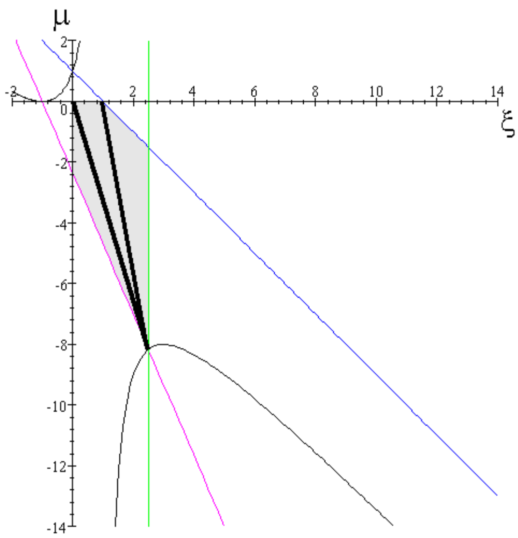

Figure 6: The stability area for the OLG model with adaptive expectations and the segment joining the Origin with the intersection point between the Neimark-Hopf and Period-Doubling bifurcation lines.

on the boundary is ruled out by facts 2., 3., and 4.. So the result follows (See figure (6) for a pictorial representation). ■

The Lemma we have just proved makes it possible to identify the stability area for the adaptive model when we consider the possiblity of changing the learning parameter α. This is done in the following Proposition.

Proposition 11 Given the model AM, for all the couple $(\xi, \mu) \in \mathbb{R}^2$ such that

i) either $0 < \xi \leq 1$ and $\mu < 0$,

ii) or $\xi > 1$ and $\frac{(1+\xi)^2}{1-\xi} < \mu < 1 - \xi$,

there exists an $\alpha \in (0, 1)$ such that the system is locally stable.

Proof. It is sufficient to show that all the points belonging to the curve of equation $\mu = \frac{(1+\xi)^2}{1-\xi}$ can be obtained as the intersection point of the Neimark-Hopf and Period-Doubling bifurcation lines by choosing an adequate value of α. Indeed we have

$$\begin{cases} \xi = \frac{1}{(1-\alpha)} \\ \mu = -\frac{(2-\alpha)(1+\xi)}{\alpha} \end{cases} \Rightarrow \begin{cases} \alpha = \frac{\xi-1}{\xi} \\ \mu = -\frac{\left(2-\frac{\xi-1}{\xi}\right)(1+\xi)}{\frac{\xi-1}{\xi}} \end{cases} \Rightarrow \begin{cases} \alpha = \frac{\xi-1}{\xi} \\ \mu = -\frac{(1+\xi)^2}{\xi-1} \end{cases} \qquad (63)$$

and for $\xi > 0$ we have $\alpha \in (0, 1)$. The result follows by Lemma 3 and 4. ■

The curve $\mu = -\frac{(1+\xi)^2}{\xi-1}$ (an hyperbola) is represented in figure (7) together with its asynthotes $\xi = 1$ and $\mu = -\xi - 3$.

3.4.1 A note on the parameter space

In the previous paragraph we have set, with equations (57), a relation between trace and determinant of the Jacobian matrix of the system and the parameters (s_w, s_r, f'', w') with respect to which we like to infer some information about how they affect the system stability. Now we want to better understand such a relation. First of all, we observe that (58) is a one to one affine map from the (ξ, μ)-plane into the (tr, det)-plane, whose matrix representation is

$$\begin{pmatrix} det \\ tr \end{pmatrix} = \begin{pmatrix} 0 & 1-\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \mu \end{pmatrix} + \begin{pmatrix} 0 \\ 1-\alpha \end{pmatrix}. \qquad (64)$$

The inverse map is

$$\begin{pmatrix} \xi \\ \mu \end{pmatrix} = \begin{pmatrix} \frac{1}{(1-\alpha)} & 0 \\ -\frac{1}{\alpha(1-\alpha)} & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} det \\ tr \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1-\alpha}{\alpha} \end{pmatrix}; \qquad (65)$$

that is

$$\begin{cases} \xi = \frac{1}{(1-\alpha)} \det \\ \mu = -\frac{1}{\alpha(1-\alpha)} \det + \frac{1}{\alpha} \mathrm{tr} - \frac{1-\alpha}{\alpha} \end{cases} \tag{66}$$

Now, remember the stability conditions given with respect to trace and determinant of J_{AM} defined in equations (59 and 61). In figure (8) is represented the stability region of parameter in the trace-determinant plane. All the lines and curves are named with small letters and all intersection points are named with capital ones. The same situation is represented, with respect to the variables (ξ, μ), in figure (9)

Let's recall the interdependence among parameters

$$\begin{aligned} \xi &\triangleq \frac{s_w w'}{1+n} \\ \mu &\triangleq \frac{s_r f''}{1+n} \end{aligned} \tag{67}$$

It would be interesting to analyze the link between the pair (ξ, μ) and the set of technology and preference specifications, that is $(s_w,\ s_r,\ f'',\ w')$. If we imagine to set the value of two of them than the situation can be summarized through either a linear or hyperbolic relationship, whereas willing to consider a wider set of cases the situation turns out to be more complex and it certainly deserves deeper analysis.

3.5 Rational vs adaptive vs myopic expectations: a comparative analysis

In this subsection we compare the dynamics under rational, adaptive and myopic expectations. First we need to determine the stability conditions for the rational and the myopic cases.

As regards the rational expectations model recall that the dynamics is determined by the implicit map

$$k_{t+1} = \frac{s\left[f\left(k_t\right) - k_t f'\left(k_t\right), f'\left(k_{t+1}\right)\right]}{1+n} \tag{68}$$

Under the assumption (23) of Proposition 1, there exists a single valued function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $k_{t+1} = \phi\left(k_t\right)$ and for the implicit function theorem in the steady state $\bar{k}$ we obtain

$$\phi'\left(\bar{k}\right) = \frac{s_w\left[w(\bar{k}), f'\left(\bar{k}\right)\right] w'\left(\bar{k}\right)}{1+n-s_r\left[w(\bar{k}), f'\left(\bar{k}\right)\right] f''\left(\bar{k}\right)} \tag{69}$$

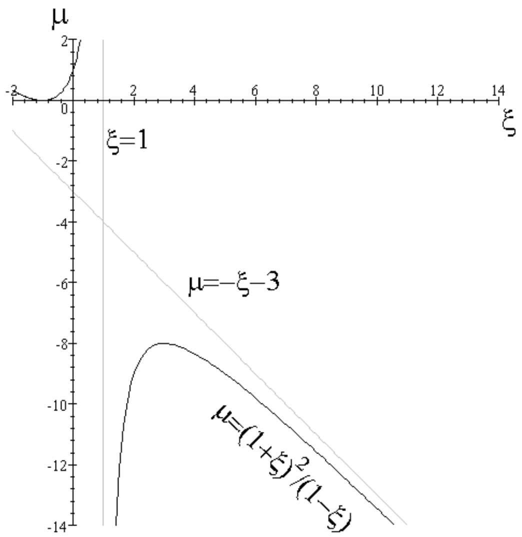

Figure 7: The hyperbola obtained as the intersection points of the Neimark-Hopf and Period-Doubling bifurcation lines by varying the value of α in $(0, 1)$.

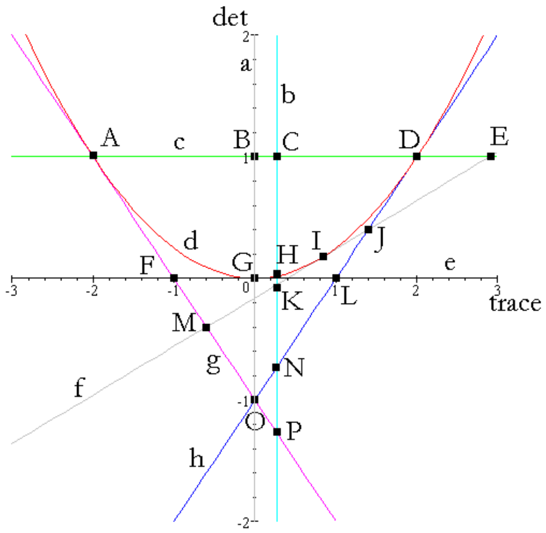

Figure 8: The bifurcation diagram in the trace-determinant plane.

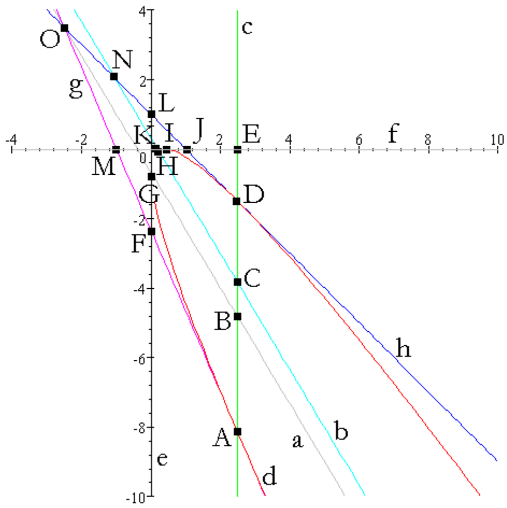

Figure 9: The bifurcation diagram in the (x, μ) plane.

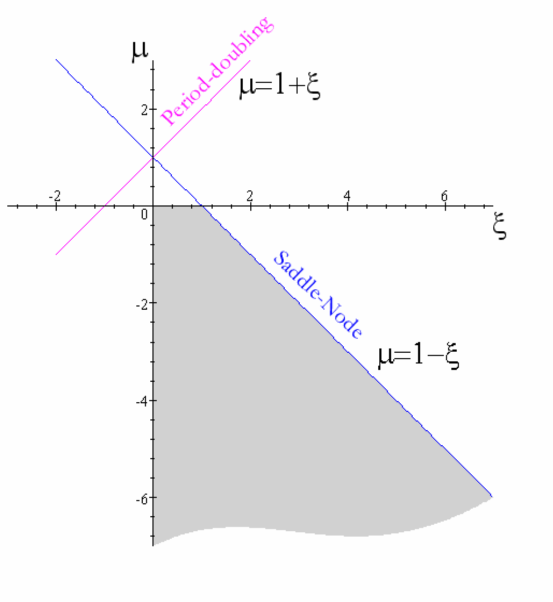

Figure 10: The stability region for the model with perfect foresight

a. The model with A is locally stable and the model with M is not.

b. The model with M is locally stable and the model with A is not.

c. Both the model with A and the model with M are locally stable.

4. All the three models can loose stability through a Saddle-Node bifurcation which arise always on the line $\mu = -\xi + 1$.

5. Both the model with A and the model with M can loose stability through a Period-doubling bifurcation, which arise on the line $\mu = -\frac{(2-\alpha)(1+\xi)}{\alpha}$ *in the model with A and on the line* $\mu = -\xi - 1$ *in the model with M.*

6. Only the model with A can experiment a Neimark-Hopf bifurcation which arise on the line $\xi = \frac{1}{(1-\alpha)}$.

Proof. It follows immediately from the stability conditions (60), (70) and (73) ■

As we have seen in Proposition 11, in the model with adaptive expectation, if we consider the possiblity for the parameter α to be adequately changed, the stability region can be enlarged to incorporate completely the stability region of the model with Myopic expectations. We have the following result.

Proposition 13 *Let* $(\bar{k}, \bar{r})$ *a stationary steady state for the map*

$$\begin{cases} k_{t+1} = \frac{s\left[f(k_t) - k_t f'(k_t), r^e_{t+1}\right]}{1+n} \\ r_{t+1} = E\left(k_t, k_{t+1}, r_t\right) \end{cases} \tag{75}$$

where $E(\cdot)$ *could be specified alternatively as the rational (R), adaptive (A) and myopic (M) expectations function. Suppose that the parameter* α *of the adaptive case could be changed in order to obtain better performances. Then:*

The model with R is locally stable

$\Downarrow$

The model with A is locally stable

$\Downarrow$

The model with M is locally stable

Proof. It follows immediately from the stability conditions (60), (70) and (73) and from Proposition 11 ■

The situation is represented in figure (12); in dark gray is shown the stability region for the myopic model, the sum of dark and medium gray gives the stability region for the adaptive model whereas the union of all gray areas represents the set of parameters which grants local stability for the model with perfect foresight.

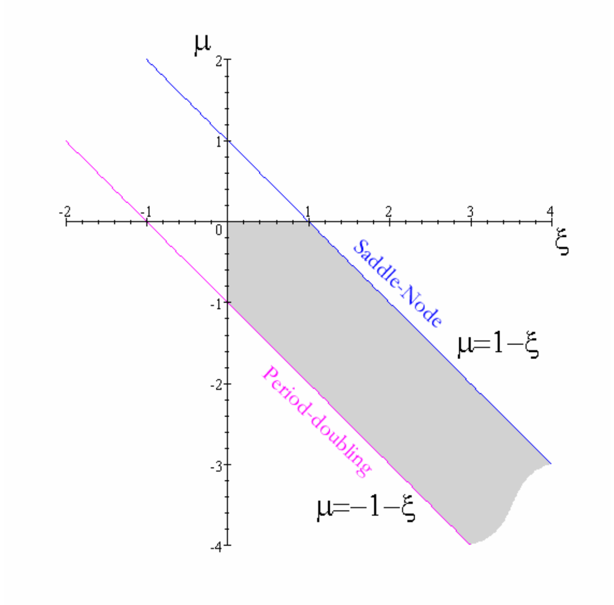

Figure 11: The stability region for the model with myopic expectations

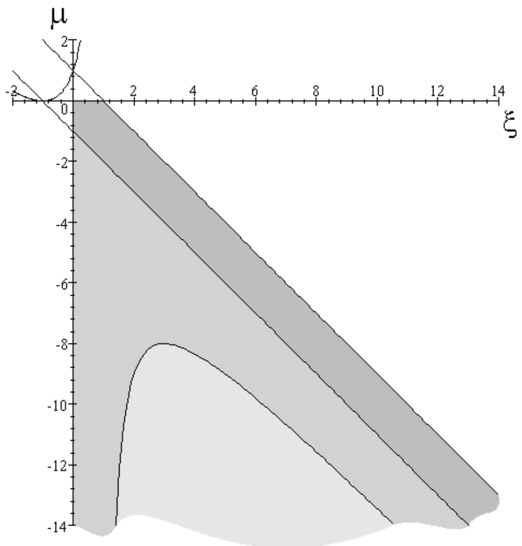

Figure 12: The comparison among global stability regions

4 Conclusions

In this paper we have dealt with an Overlapping Generations Model with production as proposed by Galor and Ryder in 1989 [6]. We have studied this model under three diverse assumptions about agents rationality. For the cases of rational and myopic expectations we have summarized the results obtained by Galor and Ryder [6] and Michel and de la Croix [8]. We have determined a uniqueness condition for stationary steady states in the model with perfect foresight which impose conditions only on the second derivative of the production function (whereas the one considered in [6] required assumptions on the third derivative too). Such condition results to be more restrictive than the one developed in [8] for the model with myopic expectations which, due to the correspondence among steady states of the three models, could be considered as an alternative. Further, we have completely developed the analysis of the model under adaptive expectations. We have derived stability conditions and determined the bifurcation diagram in all the three cases. From the comparison it results that stability conditions for the case with rational expectations are less restrictive than for both adaptive and myopic ones. We have shown that, differently from what happens in the OLG model of pure exchange, the adaptive expectations do not improve local stability performances of the model with respect to myopic expectations (see Barucci 2000 [3] for an analysis of this case); this is due to the fact that in our two-dimensional model a Neimark-Hopf bifurcation could arise, cutting off part of the parameter space which results to be stable in the myopic case.

5 Appendix

Proof of Lemma 1 Define

$$g(s;w,r) \triangleq u\left[w-s,(1+r-\delta)\,s\right], \tag{76}$$

then problem (16) can be restated as follows

$$s\,(w,r) = \arg\max_{s\in[0,w]} g(s;w,r). \tag{77}$$

Since $g(s;w,r)$ is nothing but the restriction of $u\,(c^1,c^2)$ to a convex subset of $\mathbb{R}^2_+$, namely the segment

$$\gamma \triangleq \left\{\left(c^1,c^2\right)\in\mathbb{R}^2 : \left(c^1,c^2\right) = (w,0)+s(-1,1+r-\delta), \quad 0\le s\le w\right\}, \tag{78}$$

it follows that g is strictly quasiconcave on γ. The latter property of g implies that $s\,(w,r)$ is a single valued function for each $(w,r)\in\mathbb{R}^2_{++}$. Moreover, Assumption A5) implies that problem (77) has no corner solutions. As a result, the first order conditions (F.O.C.) are necessary and sufficient for optimality of g and the function s defined in (77) is single valued. The F.O.C. of g are

$$F(s,w,r)=0, \tag{79}$$

where

$$\begin{aligned} F(s,w,r) &\triangleq \frac{d}{ds}g(s;w,r) \qquad (80)\\ &= (1+r-\delta)\,u_2\left[w-s,(1+r-\delta)\,s\right]-u_1\left[w-s,(1+r-\delta)\,s\right]. \end{aligned}$$

Assumptions A1), A3), A5) and the compactness of the feasible set γ imply that (79) does have a unique solution for each $(w,r)\in\mathbb{R}^2_{++}$. The partial derivatives $F_s(s,w,r)$ and $F_w(s,w,r)$ are

$$\begin{aligned} F_s(s,w,r) &= u_{11}-(1+r-\delta)\,u_{12}-(1+r-\delta)\,u_{21}+(1+r-\delta)^2\,u_{22} \qquad (81)\\ &= \left\langle(-1,1+r-\delta)\,H_u\left(w-s,(1+r-\delta)\,s\right),(-1,1+r-\delta)\right\rangle, \end{aligned}$$

$$F_w(s,w,r)=(1+r-\delta)\,u_{21}-u_{11}. \tag{82}$$

Assumption A3) implies $F_s(s,w,r)<0$ (see [2], Theorem 3.26, p. 78) and the implicit function theorem apply. It follows that equation (79) implicitly define a function $s=s\,(w,r)$ whose derivative with respect to the first argument, $s_w\,(w,r)$, is

$$s_w\,(w,r) = -\frac{F_w(s,w,r)}{F_s(s,w,r)}. \tag{83}$$

From (79) we obtain, thanks to assumption A2), $(1+r-\delta) = \frac{u_1}{u_2}$, which, together with assumption A4), yields

$$F_w(s,w,r) = \frac{u_1}{u_2}u_{21} - u_{11} > 0, \tag{84}$$

hence the desired conclusion. ■

Proof of Lemma 2 We proceed as we do in Proof of Lemma 1. Once again, the implicit function theorem yields (see Proof of Lemma 1 for notation)

$$s_r\,(w,r) = -\frac{F_r(s,w,r)}{F_s(s,w,r)}, \tag{85}$$

where

$$F_r(s,w,r) = u_2 + (1+r-\delta)\,u_{22}s - u_{12}s. \tag{86}$$

From (79) we get

$$(1+r-\delta) = \frac{u_1}{u_2} \quad \text{and} \quad u_2 = \frac{u_1}{(1+r-\delta)}. \tag{87}$$

If we substitute (87) into (86) we obtain

$$F_r(s,w,r) = \frac{u_1}{(1+r-\delta)} + \frac{u_1}{u_2}u_{22}s - u_{12}s. \tag{88}$$

Finally, $s_r\,(w,r) \geq 0$ if and only if $F_r(s,w,r) \geq 0$, that is,

$$s_r\,(w,r) \geq 0 \Longleftrightarrow u_1u_2 \geq (u_2u_{12} - u_1u_{22})\,(1+r-\delta)\,s, \tag{89}$$

which completes the proof. ■

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