Expectations of learning agents and stability of perfect foresight equilibria in discrete time dynamic economic models.

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October 1999.

1 Introduction

We deal with bounded rationality economic models and with some points that have attracted criticism in the literature. First, whichever the adaptive scheme used by agents in updating their expectations (recursive least square, Bayesian learning, learning through neural networks or genetic algorithms and so forth) it is typically chosen *ad hoc*. Second, the frequent arising of puzzling outcomes such as non-perfect foresight cycles seems to clash with the minimal requirement in terms of the agents' rational capacities that it is reasonable to assume.

These two seemingly uncorrelated problems can be found, in a vast class of models, the consequence of one another. As we shall see, convergence toward non-perfect foresight cycles can be caused by the fact that agents use a fixed, independent from the context rule to update their expectations. We show that, ascribing to the agents the possibility of choosing among alternatives the mechanism generating their forecasts, the system's stability increases and the non-perfect foresight cycles can be ruled out.

Gerard Fuchs attacked this problem in a 1979 paper [8], considering a model of economic dynamics whose evolution depended crucially on agent's predictions about future values of some significant of the state x describing the system. The forecasts are obtained as values of an "expectation function" ($E \in \mathbb{E}$) defined on a given set of the available information. The expectation function of an agent summarizes her (implicit or explicit) point of view on the economic system. Fuchs considered models with at least one stationary state x^* and studied its asymptotic stability; particularly let $\mathcal{E}(x^*) \subset \mathbb{E}$

the expectation functions consistent with x^* being an attractor (at least locally). Consider then a set of more sophisticated models in which agents can modify their expectation function, i.e. their "vision of things". Such updating usually follows the acquisition of new information, and exploits an *"error learning rule"* (ELR); therefore the model defines a dynamic with respect to x *and* $E \in \mathbb{E}$, so that the state of the system is given by a couple $y_t = (x_{t,} E_t)$. If more skilled agents give rise to economic system that are more stable, as it is claimed by a classical intuition, one naturally expects equilibrium convergence to occur more often than in models that have a fixed expectations updating rule. In the language of Fuchs we can expect that, given $\mathcal{E}(x^*)$, there is a set $\mathcal{L}$ of $ELRs$ generating a dynamic for which:

a) trajectories $\{y_t\}$ starting from a $y_0 = (x_0, E_0)$, $E_0 \in \mathcal{E}$ and x_0 in a neighbourhood of x^*, converge to $(x^*, \mathcal{E})$;

b) there are expectations functions E_0, $E_0 \notin \mathcal{E}$ and $y_0 = (x_0, E_0)$, with x_0 in a appropriate neighbourhood of x^*, generating trajectories that still converge to $(x^*, \mathcal{E})$.

Basically (a) says that with any E_0 in $\mathcal{E}$, the equilibrium x^* stays attracting as with fixed E; (b) renders the idea that learning can make a situation of instability of the equilibrium (under static forecasting rule) into one compatible with convergence to self-fulfilling expectations.

Fuchs requires that the set $\mathcal{L}$ have a significant dimension in topological terms.

In details, [8] works with a pure-exchange overlapping generations economy, with a constant population of heterogeneous agents (with respect to expectations). At time t the forecasts on time $t+1$, prices x^e_{t+1}, depend on current prices and on a given sequence $\mathbf{x}$ of prices observed in the past. To keep the matter simple suppose we restrict attention to a set of expectations function parametrized on $\alpha \in \mathbb{R}$. So $E(\cdot) = E(\alpha, \cdot)$. We require a rationality and a technical condition. The rationality requirement is that if the sequence $\mathbf{x}$ is stationary: $x_t = x^*$, (in which case we write $\mathbf{x} = \mathbf{x}^*$), then $E(\alpha, \mathbf{x}^*) = x^*\ \forall \alpha \in \mathbb{R}$. The technical condition is that there be a bijection between the coefficient α and the functions E. Take the set of continuously differentiable $ELRs$ such that

$$\alpha_{t+1} = G(\alpha_t, x_t - x^e_t)$$

with

$$G(\alpha, 0) = \alpha \tag{1}$$

In this situation we have the following:

Proposition 1 (Fuchs 79) *Let x^* be a stationary temporary equilibrium, $\mathcal{E}(x^*)$ the set of expectations functions for which x^* is asymptotically stable. Consider on ELR the τ_1 topology . Let $\mathcal{L} \subset ELR$ be such that for all elements of $\mathcal{L}$ we can find a neighbourhood U of x^* such that:*

a) the trajectories $\{y_t\}$ determined by the initial condition $y_0 = (x_0, \alpha_0)$ with $x_0 \in U$ and $\alpha_0 \in \mathcal{E}(x^*)$, converge to $(x^*, \mathcal{E})$;

b) there exist trajectories $\{y_t\}$ determined by initial conditions $y_0 = (x_0, \alpha_0)$ with $x_0 \in U$ and $\alpha_0 \notin \mathcal{E}(x^*)$, that still converge to $(x^*, \mathcal{E})$.

Then $\mathcal{L}$ is a closed set with an empty interior (in the τ_1 topology).

This result (a slightly simplified version of the original) states that learning devices of the type described by Fuchs enhance the stability of the system in a negligible set of cases. This conclusions draws heavily on assumption (1), which appears natural in this scheme, with the law E representing the agents' "vision of things". (1) simply says that, in case of perfect anticipation of the future prices, agents would deduce that their theory is right and would therefore leave it unchanged to formulate forecasts in the next period. In fact this interpretation, in most models described in the literature, is not easily convincing. as we shall see, replacing (1) with a different assumption, this result can be radically changed.

2 A general framework

We start from a very general class of models in which a vector of variables at time s, $\mathbf{x}_s \in X$, is a function of agents' expectations on the value the vector will take at time t, $\mathbf{x}_t^e \in X$, with $t = s + z$, z an integer and $X \subseteq \mathbb{R}^m$ the set of economically significant values of $\mathbf{x}$:

$$\mathbf{x}_s = F(\mathbf{x}_t^e) \tag{2}$$

In order to maintain the scheme described in [8], we are interested in economic models in which agents perform their forecasts relying on the vector of past values of the observed variable. It is well known that adaptive expectations means that the forecasts are built through a weighted average, with geometrically declining weights, of all past observed values. Indeed, given the vector of (infinite) past observations $\mathbf{x} = (x_s, x_{s-1}, ...)$ and $0 < \alpha < 1$, we have

$$x_{t+1}^e = \sum_{j=0}^{\infty} \alpha (1-\alpha)^j x_{s-j} = x_t^e + \alpha (x_s - x_t^e)$$

In such a situation we implicitly assume that the Economy has no beginning date, in order to have an infinite number of past data. A more interesting way to obtain the same model is that of considering the limit map of a learning mechanism based on Mann Iterations. In this case the Economy is supposed to have a beginning date and the vector of past data contains a finite number of elements. To obtain more insights about this point of view see [12, 2, 1, 4, 3, 5]

Hence, assume that the agents are characterised by bounded rationality and that they update their expectations through an adaptive learning rule

$$\mathbf{x}^e_{t+1} = \mathbf{x}^e_t + A(\mathbf{x}_s - \mathbf{x}^e_t) \tag{3}$$

where A is a diagonal matrix with $a_{i,i} = \alpha_i \in \mathcal{A}$ and $\mathcal{A}$ is the set of economically meaningful values of α (typically $[0,1]$), so, substituting (2) in (3)

$$\mathbf{x}^e_{t+1} = E\left(\mathbf{x}_t, \boldsymbol{\alpha}\right) = \mathbf{x}^e_t + A\left[F(\mathbf{x}^e_t) - \mathbf{x}^e_t\right] \tag{4}$$

Accordingly with the foregoing discussion, now we assume that the agents, on the basis of past errors, update the coefficients α_i, which represent the weight ascribed to older data when formulating the new expectations. For a simpler exposition, let us consider the scalar case $\mathbf{x}_s = x_s \in \mathbb{R}$, so that

$$x^e_{t+1} = x^e_t + \alpha\left[F(x^e_t) - x^e_t\right] \tag{5}$$

We will return back to the general case in section 4. It is worth noting that this simplification does not deprive the model of its significance; given $s = t-1$ we obtain

$$\begin{aligned} x_t &= F(x^e_{t+1}) \\ x^e_{t+1} &= x^e_t + \alpha(x_{t-1} - x^e_t) \end{aligned}$$

i.e. a classic forward looking model of the kind that, for example, is to be obtained in simple O.L.G. models under bounded rationality. Whereas, given $s = t$ we have

$$\begin{aligned} x_t &= F(x^e_t) \\ x^e_{t+1} &= x^e_t + \alpha(x_t - x^e_t) \end{aligned}$$

which we could obtain in a Cobweb model with adaptive expectations.

Now, consider a model in which, at time t, traders update their expectations on the basis of a weighted average of past data and weights geometrically decreasing with ratio α_t, and suppose that such a ratio is, at the same time, updated through a general law g which determines its dynamic:

$$x_s = F(x^e_t) \tag{6a}$$

$$x^e_{t+1} = x^e_t + \alpha_t(x_s - x^e_t) \tag{6b}$$

$$\alpha_{t+1} = g(x^e_t, x^e_{t+1}, x_s, x_{s+1}, \alpha_t) \tag{6c}$$

Remark that this is a more general case with respect to that considered in [8], as the ELR represent only a subset of possible specifications of the law g.

We analyse the dynamical system, obtained from (1a), (1b) and (1c) substituting for x_t and x_{t-1} in (1b) and (1c) and for x^e_{t+1} in (1c):

$$(P): \begin{cases} x^e_{t+1} = E\left(x^e_t, \alpha_t\right) = x^e_t + \alpha_t(F(x^e_t) - x^e_t) \\ \alpha_{t+1} = G(x^e_t, \alpha_t) \end{cases} \tag{7}$$

Notice that in order to have a steady state with $\alpha^* \neq 0$ then x^* must be a fixed point of the map F. The Jacobian matrix for (7) evaluated at (x^*, α^*) is:

$$J(x^*, \alpha^*) = \begin{pmatrix} 1 + \alpha^*[F'(x^*) - 1] & 0 \\ \frac{\partial G}{\partial x^e_t}(x^*, \alpha^*) & \frac{\partial G}{\partial \alpha_t}(x^*, \alpha^*) \end{pmatrix} \tag{8}$$

whose eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 + \alpha^*[F'(x^*) - 1] \\ \lambda_2 &= \frac{\partial G}{\partial \alpha_t}(x^*, \alpha^*) \end{aligned}$$

Hence, the steady state is locally stable when $\alpha^* > 0$ if

$$F'(x^*) < 1 \qquad \text{e} \qquad \alpha^* < \frac{2}{1 - F'(x^*)} \tag{9}$$

$$-1 < \frac{\partial G}{\partial \alpha_t}(x^*, \alpha^*) < 1 \tag{10}$$

Notice that (9) is the same as in a model with a constant $\alpha \neq 0$ (see [1] for an analysis of this case). Therefore we can expect more stringent stability conditions (embedded in (10)) due to the introduction of the endogenous mechanism generating α_t.

It can be interesting to express condition (10) in terms of the original function g defined in (1c). Evaluating the total derivative of g with respect to α_t we obtain:

$$\begin{aligned} & \frac{\partial G}{\partial \alpha_t}(x^e_t, \alpha_t) = \frac{dg}{d\alpha_t}(x^e_t, x^e_{t+1}, x_s, x_{s+1}, \alpha_t) = \\ = \; & \frac{dg}{d\alpha_t}(x^e_t, x^e_{t+1}, F(x^e_t), F(x^e_{t+1}), \alpha_t) \equiv \frac{dg}{d\alpha_t}(X) \\ = \; & \frac{\partial g}{\partial x^e_{t+1}}(X) \frac{\partial x^e_{t+1}}{\partial \alpha_t}(x^e_t, \alpha_t) + \frac{\partial g}{\partial F(x^e_{t+1})}(X) \frac{\partial F(x^e_{t+1})}{\partial \alpha_t}(x^e_t, \alpha_t) + \frac{\partial g}{\partial \alpha_t}(X) \\ = \; & \frac{\partial g}{\partial x^e_{t+1}}(X)[F(x^e_t) - x^e_t)] + \frac{\partial g}{\partial F(x^e_{t+1})}(X) F'(x^e_{t+1})[F(x^e_t) - x^e_t)] + \frac{\partial g}{\partial \alpha_t}(X) \end{aligned}$$

Therefore at the steady state we end up with:

$$\frac{\partial G}{\partial \alpha_t}(x^*, \alpha^*) = \frac{\partial g}{\partial \alpha_t}(x^*, x^*, x^*, x^*, \alpha^*) \tag{11}$$

A consequence of this fact is that, with regard to the local stability properties of the model, different specifications of the functional form g will bring about distinct outcomes only inasmuch as they have different partial derivatives with respect to α_t. Moreover, if the learning step is made endogenous through a law which does not autonomously depend on α (i.e. if $\frac{\partial g}{\partial \alpha_t} = 0$), local stability towards a given steady state (x^*, α^*) occurs under the same circumstances as in the exogenous case (constant α, as in [1]).

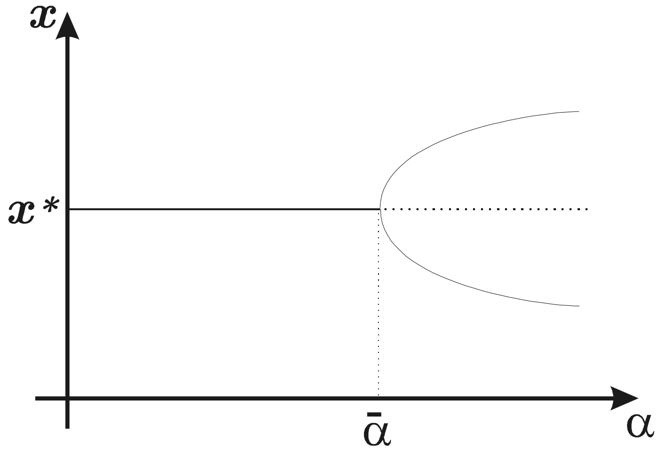

Figure 1: A Period Doubling bifurcation diagram.

It is interesting to compare what happens in the present context with the unidimensional system that has a fixed α. In the latter case we can refer to the diagram of Figure (1) (which, with $0 < \alpha < 1$, can be sketched if $F'(x^*) < -1$), where beyond a critical point $\bar{\alpha}$ the steady state looses stability in favour of a cycle of period 2 (flip bifurcation). Two points are worth emphasizing: first, as it can be easily shown, they are *not* perfect foresight cycles, in the sense that they imply agents' systematic forecasting errors; second, these cycles are significant because they attract any point however close to the steady state. This means that agents in the model are driven to "learn" cycles along which their expectations are not self-fulfilling; in fact, such an outcome entails that the learning process halts, which is clearly an undesirable feature of the model.

What we do in the next Section is to show that, as soon as an endogenous dynamics for α is introduced, the situation radically changes: in particular, for whatever initial value of α, starting from beliefs which are near to the steady state x^*, agents will be able to correct their expectations towards x^*: so there is actual learning up to a situation of self-fulfilling beliefs.

3 Some facts from global analysis for a class of G functions

We try to stick to as general a framework as possible. Let us rewrite the system analysed in Section 2:

$$(P): \begin{cases} x_{t+1}^e = E\left(x_t^e, \alpha_t\right) = x_t^e + \alpha_t(F(x_t^e) - x_t^e) \\ \alpha_{t+1} = G(x_t^e, \alpha_t) \end{cases}$$

We are going to need the following assumptions:

[A1] (x^*, α^*) is a locally unique, asymptotically stable[1] fixed point for (P);

[A2] F, G are continuously differentiable;

[A3] $\exists \gamma \in (0,1) : |\alpha^* - G(x^*, \alpha_t)| < \gamma \, |\alpha^* - \alpha_t| \quad \forall \alpha_t \neq \alpha^*$.

[A1] defines the object of our analysis; [A2] is a standard regularity requirement. [A3] says that if the state variable x is at an equilibrium value, then the learning step variable must tend to its equilibrium value too.

Clearly [A3] is the counterpart of (1). In our interpretation α serves as a tool to provide better forecasts: this goal does not really make sense when the system is in x^*, so it is plausible to ask that the dynamics of α should reach a state of rest as well. The fact that we require uniqueness for this equilibrium and so that we rule out the indeterminacy that shows up in Fuchs [8], depends on the class of models we are interesting in and on their economic interpretation.

Theorem 1 is the main result of this section. Its proof requires the following Lemma.

In what follows, for the sake of simplicity, we could consider $x, \alpha \in \mathbb{R}$. But it would be misleading from an economic point of view. On the other

[1] A closed subset C of the state space X, invariant for a map P, is *Lyapunov stable* if for any neighbourhood V of C there is a neighbourhood $V' \subset V$ of C such that $P(V') \subset V$. Further, C is *asymptoticlly stable* if it is Lyapunov stable and it has an open basin of attraction $\mathfrak{B}(A)$. $\mathfrak{B}(A)$ is the set of $x \in X$ such that $P^n(x) \to A$ as $n \to \infty$. In this case, if V is su that its closure is contained in $\mathfrak{B}(A)$, we have $A = \bigcap_{n>0} P^n(V)$. For more details see [7, 9, 13, 14]

hand we could consider $x \in X$ e $\alpha \in \mathcal{A}$. In this case, writing that x belongs to a given set I or that α belongs to another set J, we will mean $x \in (I \cap X)$ e $\alpha \in (J \cap \mathcal{A})$respectively. Obviously, it is necessary that X is such that $E : X \times \mathcal{A} \to X$ is granted.

Lemma 1 *For the system* (P), *given [A1], [A2] and [A3], for any* $\hat{\alpha}_0 \neq \alpha^*$ *there are two monotone sequences* $\{\hat{\alpha}_n\}, \hat{\alpha}_n \to +\infty$, $\{\hat{\beta}_n\}, \hat{\beta}_n \to -\infty$, $\hat{\alpha}_n > \hat{\beta}_n$ *and a positive, non-increasing sequence* $\{\delta_n\}$, *such that*

$$(x_t, \alpha_t) \in I_{\delta_{n+1}}(x^*) \times \left\{[\hat{\alpha}_n, \hat{\alpha}_{n+1}] \cup [\hat{\beta}_{n+1}, \hat{\beta}_n]\right\}$$
$$\Downarrow$$
$$(x_{t+1}, \alpha_{t+1}) \in I_{\delta_n}(x^*) \times (\hat{\beta}_n, \hat{\alpha}_n)$$

Proof.

Suppose, without loss of generality, $\hat{\alpha}_0 > \alpha^*$ and let γ be such that [A3] applies to all $\alpha_t : |\alpha_t - \alpha^*| > |\alpha_0 - \alpha^*|$. Let $\hat{\beta}_0 = 2\alpha^* - \hat{\alpha}_0$, that is the symmetric point of $\hat{\alpha}_0$ with respect to α^*. Consider the sequences

$$\hat{\alpha}_{n+1} = \frac{\hat{\alpha}_n}{\gamma} - \frac{1-\gamma}{\gamma}\alpha^* \tag{12}$$

$$\hat{\beta}_{n+1} = \frac{\hat{\beta}_n}{\gamma} - \frac{1-\gamma}{\gamma}\alpha^* \tag{13}$$

It can be easily seen that $\{\hat{\alpha}_n\}$ is increasing and tends to $+\infty$, whereas $\{\hat{\beta}_n\}$ is decreasing and tends to $-\infty$. Let $[\hat{\alpha}_n, \hat{\alpha}_{n+1}] \cup [\hat{\beta}_{n+1}, \hat{\beta}_n] \equiv K_n$. Observe that for all $\alpha_t \in K_n$ we have

$$\begin{aligned} |\alpha^* - G(x^*, \alpha_t)| &= |\alpha^* - \alpha_{t+1}| < \\ &< \gamma |\alpha^* - \alpha_t| \leq \gamma |\alpha^* - \hat{\alpha}_{n+1}| = |\alpha^* - \hat{\alpha}_n| \end{aligned}$$

using [A3] and (12). By continuity, given any $\bar{\alpha} \in K_n$ we can find $\delta_{\bar{\alpha}} > 0$ such that

$$|\alpha^* - G(x_t^e, \alpha_t)| < |\alpha^* - \hat{\alpha}_n| \tag{14}$$

for all $(x_t^e, \alpha_t) \in I_{\delta_{\bar{\alpha}}}(x^*, \bar{\alpha})$. Actually, we can choose δ'_{n+1} so that $\delta_{\bar{\alpha}} \geq \delta'_{n+1} > 0$ for all $\bar{\alpha} \in K_n$: indeed, suppose we could not; then there would exist a sequence $\{x_j^e, \alpha_j\} \to (x^*, \tilde{\alpha})$, $\tilde{\alpha} \in K_n$ for which

$$\left|\alpha^* - G(x_j^e, \alpha_j)\right| \geq \gamma |\alpha^* - \alpha_j| \qquad j \in \mathbb{N}$$

But this would contradict (14). Finally, for any $\delta_n > 0$ we can find $0 < \delta_{n+1} \leq \delta'_{n+1}$ such that for all $(x_t, \alpha_t) \in I_{\delta_{n+1}}(x^*) \times K_n$

$$\left|x_{t+1}^e - x^*\right| \leq |x_t^e - x^*| + |\alpha_t| \, |F(x_t^e) - x_t^e| \leq |x_t^e - x^*| + |\hat{\alpha}_{n+1}| \, |F(x_t^e) - x_t^e| < \delta_n \tag{15}$$

because, given that F is continuous and $F(x^*) = x^*$ a small enough δ_{n+1} will make $|F(x_t^e) - x_t^e|$ arbitrarily small as well. See Figure 2. Notice that, using (15), we only need to fix $\delta_0 > 0$ to retrieve recursively the entire sequence $\{\delta_n\}$. (14) and (15) show that the two sequences have the required properties. This concludes the proof. ■

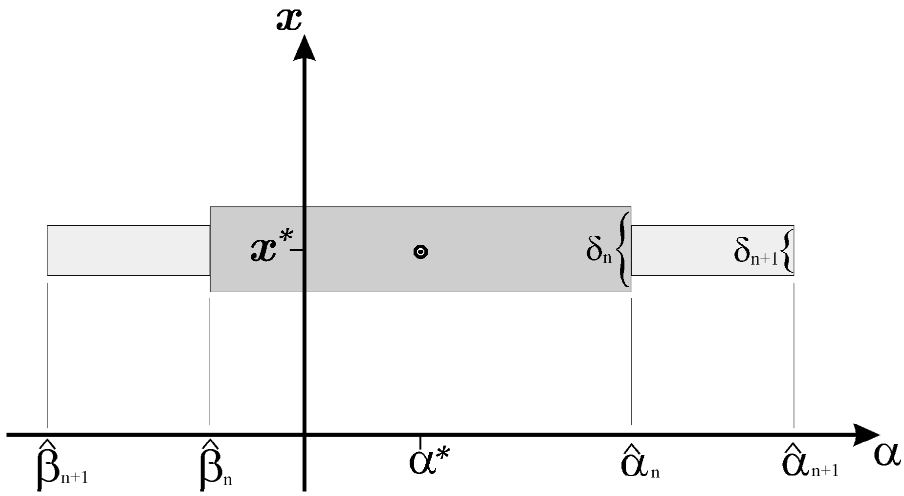

Figure 2: The map takes, in one step, points from the light grey region into the dark rectangle.

Theorem 1 *Under assumptions [A1], [A2], [A3], for all* $\alpha \in \mathbb{R}$ *there is a positive* δ *(depending on* α*) such that*

$$I_\delta(x^*) \times \{\alpha\} \subset \mathfrak{B}(x^*, \alpha^*) \tag{16}$$

$\mathfrak{B}(x^*, \alpha^*)$ *being the basin of attraction of* (x^*, α^*).

Proof. Thanks to [A1] we can find a square neighbourhood of (x^*, α^*) belonging to $\mathfrak{B}(x^*, \alpha^*)$: let $\hat{\delta}$ measure its side and define $\hat{\beta} = \alpha^* - \hat{\delta}/2$, $\hat{\alpha} = \alpha^* + \hat{\delta}/2$. Suppose then $\alpha \notin [\hat{\beta}, \hat{\alpha}]$. Consider $\{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$ and $\{\delta_n\}$ of Lemma **??** with $\hat{\alpha}_0 = \hat{\alpha}$, $\hat{\beta}_0 = \hat{\beta}$ and $\delta_0 = \hat{\delta}$. Then we can find n such that $\alpha \in [\hat{\alpha}_n, \hat{\alpha}_{n+1}] \cup [\hat{\beta}_{n+1}, \hat{\beta}_n]$. Then $\delta = \delta_{n+1}$ satisfies (16). ■

The interpretation of Theorem 1 is quite clear: a locally stable, unique steady state will attract points of the type (x, α) for all α, provided x is sufficiently near to x^*. We have an "eye-shaped" region which is guaranteed to be embedded in the basin of attraction of the steady state. Figure 3 provides a pictorial representation. This situation contrasts with the case

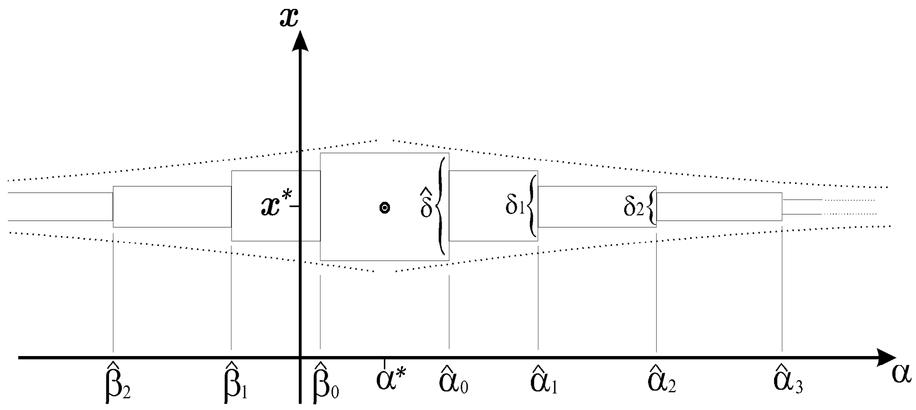

Figure 3: The "eye-shaped" region, nested in the Basin of Attraction of (x^*, α^*).

of α greater than its bifurcation value in the model with exogenous α. So, if the expectation function evolves endogenously (through the updating of the coefficient that weighs past observations) has the effect of enlarging the parameters set consistent with convergence towards the *perfect foresight* outcome, eliminating at the same time one major drawback of the original model..

Remark 1 *In order to prove Lemma 1 and then Theorem 1 assumption [A3] can be slightly relaxed, in the sense that we only require it to drag the system in the vicinity of the equilibrium. We could then simply ask that:*

$$[\text{A4}]\ \forall\delta > 0\ \exists\gamma \in (0,1) : |\alpha^* - G(x^*, \alpha_t)| < \gamma\, |\alpha^* - \alpha_t| \quad \forall\alpha_t \notin I_\delta(\alpha^*)$$

Remark 2 (A4) *, which is similar but not identical to [A3], encompasses some significant cases that we would otherwise have excluded; in particular when the force* γ *tends to* 1 *as one approaches the equilibrium. We will meet an example of this type in Section 3.1.*

From now on we label $\mathcal{G}$ the set of functions G that satisfy [A1], [A2] and [A3] whereas $\mathcal{G}^*$ is the set of Gs satisfying [A1], [A2] and [A4]; clearly $\mathcal{G} \subset \mathcal{G}^*$.

It is worth noting that all ELRs (actually a wider family $\mathcal{G}^*$) consistent with [A1], [A2] and [A4], satisfy the requirements of points 1) and 2) for Proposition 1: indeed, Theorem 1 states that, for any $\alpha > \bar{\alpha}$, for which, that is, x^* is not stable for the map (5), there is a neighbourhood U of x^* such that a learning rule $G \in \mathcal{G}^*$ gets the system (7) to converge to an equilibrium

(x^*, α^*), for any initial condition (x, α), $x \in U$. We will return on this point with Theorem 4.

Now we want to show that the set of functions $\mathcal{G}$ to which we can apply Theorem 1 is not "small", in a topological sense. We state the result here. A formal proof in a multidimensional setting can be found in Section 4.

Theorem 2 *Let x^* be a locally unique, perfect foresight stationary equilibrium for the map (2), $\mathcal{E}(x^*)$ the set of expectations functions (5), parametrized by α, for which x^* is a temporary stationary attractive hyperbolic equilibrium. Let $\mathcal{G}$ be the set of learning rules $G(x_t^e, \alpha_t)$ for which the system (P) satisfies [A1], [A2] and [A3]. Consider the set $\Gamma = \{G : \alpha_{t+1} = G(x_t^e, \alpha_t); G \in C^1\}$, of the C^1 learning rules, endowed with the topology τ_p.*

Then $\mathcal{G}$, as a subset of Γ, has a non empty interior.

3.1 A trivial example

We now present some examples in which the approach developed here can usefully be exploited. Lucas [11] contains a simple model of overlapping generations monetary economy in the Samuelson case, which we can use a starting point.

It is a pure-exchange economy with a constant population of two-period living agents endowed with a unit of perishable good in their youth period (and non when old); the stock of government-issued fiat money is constant and it is initially distributed to the old generation at time $t = 0$. Agents derive satisfaction from consumption according to:

$$U(c_t, c_{t+1}) = \sqrt{c_t} + 2c_{t+1}$$

The budget constraint is:

$$c_t + c_{t+1}\frac{q_t}{q_{t+1}} = 1$$

where $0 \le q_t = \frac{1}{p_t} \le 1$.

In equilibrium the first order conditions applied to this maximization problem lead to:

$$q_{t+1} = F(q_t) = \frac{1}{4}(1 - q_t)^{-\frac{1}{2}} q_t \tag{17}$$

Sequences that satisfy (17) are perfect foresight equilibria for the model; in particular there are two steady states:

- $q_t = 0$ is the autarchy equilibrium, in which money is worthless;

- $q_t = \frac{15}{16}$ is the monetary equilibrium.

Under perfect foresight the system will converge to $q_t = 0$; moreover all trajectories $\{q_t\}$ with $q_0 \in (0, \frac{15}{16})$ satisfy (17) for each t and constitute a whole continuum of rational expectations non-stationary equilibria. This is the classical indeterminacy puzzle of overlapping generations economies. See Geanakoplos [10] for a discussion. A commonly used way out of multiplicity is the idea of selecting an equilibrium as the outcome of the adaptive process of agents. This approach is valuable as it adds emphasis on the psychological attitude that form (optimal) decision rules, building on some learning stage. It is indeed a partial solution, though, in that the outcomes that are reached are very much the product of the particular adaptive scheme one is using, whereas there does not seem to be a universal way of deciding which scheme to adopt.

In the sequel we analyse the consequences of introducing adaptive behaviour in the model, with various examples displaying different features.

Consider the following type of expectations:

$$q^e_{t+1} = q^e_t + \alpha(q_{t-1} - q^e_t) \tag{18}$$

The possibility of regarding (18) as the limiting map of a more complex mechanism of expectation formation is investigated in some length in the next section. Substituting for $q_{t-1} = F^{-1}(q_t) \equiv Z(q_t)$ from (17) we get:

$$\begin{aligned} q^e_{t+1} &= q^e_t + \alpha(Z(q^e_t) - q^e_t) \\ Z(q^e_t) &= 4(q^e_t)^2\sqrt{4(q^e_t)^2 + 1} - 8(q^e_t)^2 \end{aligned} \tag{19}$$

It is straightforward to verify that, for all $\alpha \in (0, 1)$, the monetary equilibrium $q = \frac{15}{16}$ is stable for the dynamics in (19), whereas $q = 0$ is unstable. This follows from the fact that for the forward in time dynamics, F, $q = \frac{15}{16}$ is unstable and $q = 0$ is stable. Therefore we have a uniform behaviour for all values of $\alpha \in (0, 1)$, with no bifurcation occurring.

A different possibility is to set

$$q^e_{t+1} = \frac{1}{t}\sum_{i=0}^{t-1} q_i \tag{20}$$

This mechanism is the same as the one used in Bray [6]. It is actually a simple case, similar to (18) except for the presence of a non-constant parameter α; indeed, given (20), we can write, as in Lucas [11]:

$$q^e_{t+1} = q^e_t + \frac{1}{t}(Z(q^e_t) - q^e_t) \tag{21}$$

The possibility of embedding (21) in a more general framework is studied in the next section. As for (21), remark that it can be put in the following form:

$$\begin{cases} q^e_{t+1} = q^e_t + \alpha_t(Z(q^e_t) - q^e_t) \\ \alpha_{t+1} = G(\alpha_t) \equiv \frac{\alpha_t}{1+\alpha_t} \qquad \alpha_0 = 1 \end{cases} \tag{22}$$

which is suitable for the application of Theorem 1.

Proposition 2 *The system (22) satisfies assumptions [A1] to [A3] of Theorem 1.*

Proof. We simply list how the five conditions are met:

[A1] $(\frac{15}{16}, 0)$ is a locally stable fixed point of (22), see for instance Lucas [11] for a proof;

[A2] G and Z are clearly C^1;

[A3] given $\delta > 0$ set $\tilde{t} : \frac{1}{\tilde{t}} < \delta$ then choose any $\gamma \in (1 - \frac{1}{\tilde{t}+1}, 1)$

■

So we can actually apply Theorem 1. Observe that thanks to our result the stability argument concerning the set

$$A = (0,1) \times \left\{\frac{1}{t}\right\}_{t\in\mathbb{N}}$$

which is contained in Lucas [11], can certainly be extended to the region $A' = (0,1) \times [0,1]$.

4 The n-dimensional case

As we have announced, the result of Theorem (1) can be easily extended to the n-dimensional case.

Consider $\mathbf{x} \in \mathbb{R}^n, \boldsymbol{\alpha} \in \mathbb{R}^n$ and a forward-looking model,

$$\mathbf{x}_t = F(\mathbf{x}^e_{t+1}) = (F_1(\mathbf{x}^e_{t+1}), ..., F_n(\mathbf{x}^e_{t+1})$$

under bounded rationality and a representative agent assumption. Traders formulate expectations about each variable as a convex combination of the last forecast and observed value, $x^e_{i,t+1} = E_i(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) = x^e_{i,t} + \alpha_{i,t}(x_{i,t-1} -$

$x^e_{i,t}), i = 1...n$, where the weights $\alpha_{i,t}$ are endogenously updated through a law

$$\boldsymbol{\alpha}_{t+1} = g(\mathbf{x}^e_t, \mathbf{x}^e_{t+1}, \mathbf{x}_{t-1}, \mathbf{x}_t, \boldsymbol{\alpha}_t) = G(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) = (G_1(\mathbf{x}^e_t, \boldsymbol{\alpha}_t), ..., G_n(\mathbf{x}^e_t, \boldsymbol{\alpha}_t))$$

as in the scalar case.

Let A be the diagonal matrix with $\alpha_{i,i} = \alpha_i$, the system becomes

$$(M) \quad : \quad \begin{cases} \mathbf{x}^e_{t+1} = E\left(\mathbf{x}^e_t, \boldsymbol{\alpha}_t\right) = \mathbf{x}^e_t + A_t \left[F(\mathbf{x}^e_t) - \mathbf{x}^e_t\right] \\ \boldsymbol{\alpha}_{t+1} = G(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) \end{cases} = \tag{23}$$

$$= \begin{cases} x^e_{1,t+1} = E_1(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) = x^e_{1,t} + \alpha_{1,t}\left[F_1(\mathbf{x}^e_t) - x^e_{1,t}\right] \\ ... \\ x^e_{n,t+1} = E_n(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) = x^e_{n,t} + \alpha_{n,t}\left[F_n(\mathbf{x}^e_t) - x^e_{n,t}\right] \\ \alpha_{1,t+1} = G_1(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) \\ ... \\ \alpha_{n,t+1} = G_n(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) \end{cases} \tag{24}$$

If a vector $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ with $\boldsymbol{\alpha}^*_i \neq 0, \forall i$, is a steady state for (M), then $\mathbf{x}^*$ is a fixed point for the map $\mathbf{x}_t = F(\mathbf{x}^e_{t+1})$, that is a rational expectations equilibrium. The Jacobian matrix of (23), calculated in $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ is

$$M'\left(\mathbf{x}^*, \boldsymbol{\alpha}^*\right) = \begin{pmatrix} E'(\mathbf{x}^*, \boldsymbol{\alpha}^*) & 0 \\ G_{\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\alpha}^*) & G_{\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}^*) \end{pmatrix}$$

where E' is the Jacobian for the system (4), whereas $G_{\mathbf{x}}$ and $G_{\boldsymbol{\alpha}}$ contain the derivatives of G with respect to $\mathbf{x}$ and (respectively) $\boldsymbol{\alpha}$. The stability conditions are therefore tighter than with a constant $\boldsymbol{\alpha}$. In particular, if agents update α_i without reference to the value of α_j, $j \neq i$, the stationary state $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ for the system (23) will be locally stable if:

1. $\mathbf{x}^*$ is a locally stable equilibrium for the map (4) with $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$.
2. $-1 < \frac{\partial G_i}{\partial \alpha_i}(\mathbf{x}^*, \boldsymbol{\alpha}^*) < 1$, $i = 1...n$.

Let us now turn to the general result.

Theorem 3 *Consider the following assumptions:*

[B1] $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ is a locally unique, asymptotically stable, fixed point for (M);

[B2] $F, G \in C^1$;

[B3] $\exists \gamma \in (0,1) : \|\boldsymbol{\alpha}^* - G(\mathbf{x}^*, \boldsymbol{\alpha}_t)\| < \gamma \|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\| \quad \forall \boldsymbol{\alpha}_t \neq \boldsymbol{\alpha}^*$, where $\|\|$ is the Euclidean norm.

Then, for all $\boldsymbol{\alpha} \in \mathbb{R}^n$ there exists $\delta > 0$ (depending on $\boldsymbol{\alpha}$) such that

$$B_\delta(\mathbf{x}^*) \times \{\boldsymbol{\alpha}\} \subset \mathfrak{B}(\mathbf{x}^*, \boldsymbol{\alpha}^*) \tag{25}$$

where $B_\delta(\mathbf{x}^*)$ is a ball of radius δ centred in $\mathbf{x}^*$.

Proof.

Following the same strategy of Lemma 1 and Theorem 1 let us build, through backward induction, a sequence of neighbourhoods of the stationary state, belonging to its basin of attraction.

Thanks to [B1], there are $\delta_0, \mu_0 > 0$ such that

$$I_0(\mathbf{x}^*, \boldsymbol{\alpha}^*) = \left\{B_{\delta_0}(\mathbf{x}^*) \times B_{\mu_0}(\boldsymbol{\alpha}^*)\right\} \subset \mathfrak{B}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$$

Given $\hat{\boldsymbol{\alpha}}_0 \neq \boldsymbol{\alpha}^*$ such that $\mu_0 \geq \|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_0\|$ and γ satisfying [B3], consider the sequence of vectors

$$\hat{\boldsymbol{\alpha}}_{n+1} = \frac{\hat{\boldsymbol{\alpha}}_n - (1-\gamma)\boldsymbol{\alpha}^*}{\gamma} \tag{26}$$

We have

$$\|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_{n+1}\| = \left\|\boldsymbol{\alpha}^* - \frac{\hat{\boldsymbol{\alpha}}_n - (1-\gamma)\boldsymbol{\alpha}^*}{\gamma}\right\| = \left\|\frac{\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_n}{\gamma}\right\| = \frac{1}{\gamma}\|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_n\|$$

and therefore $\{\mu_n\} \equiv \{\|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_n\|\}$ is monotone increasing and tends to $+\infty$.

Consider now $C_n \equiv \overline{\left\{B_{\mu_{n+1}}(\boldsymbol{\alpha}^*) - B_{\mu_n}(\boldsymbol{\alpha}^*)\right\}}$, $\bar{A}$ being the closure of A. For any $\boldsymbol{\alpha}_t \in C_n$, thanks to [B3] and (26), we have

$$\begin{aligned} \|\boldsymbol{\alpha}^* - G(\mathbf{x}^*, \boldsymbol{\alpha}_t)\| &= \|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_{t+1}\| < \gamma\|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\| \leq \gamma\|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_{n+1}\| = \\ &= \|\gamma\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_n + (1-\gamma)\boldsymbol{\alpha}^*\| = \|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_n\| = \mu_n \end{aligned}$$

By continuity, for all $\boldsymbol{\alpha} \in C_n$ we can find $\delta_{\boldsymbol{\alpha}} > 0$ such that

$$\|\boldsymbol{\alpha}^* - G(\mathbf{x}^e_t, \boldsymbol{\alpha}_t)\| < \|\boldsymbol{\alpha}^* - \hat{\boldsymbol{\alpha}}_n\| = \mu_n \tag{27}$$

for any $(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) \in B_{\delta_\alpha}(\mathbf{x}^*, \boldsymbol{\alpha})$. Note that there is $\delta'_{n+1} > 0$ such that $\delta_{\boldsymbol{\alpha}} \geq \delta'_{n+1} > 0$ for all $\boldsymbol{\alpha} \in C_n$; otherwise, there would exist a sequence $\left\{\left(\mathbf{x}^e_j, \boldsymbol{\alpha}_j\right)\right\} \to \{(\mathbf{x}^*, \tilde{\boldsymbol{\alpha}})\}$ with $\tilde{\boldsymbol{\alpha}} \in C_n$ such that

$$\left\|\boldsymbol{\alpha}^* - G(\mathbf{x}^e_j, \boldsymbol{\alpha}_j)\right\| \geq \gamma\|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_j\| \qquad j \in \mathbb{N}$$

contradicting (27). Finally, given the diagonal matrix A_t with the elements of $\boldsymbol{\alpha}_t$ on the principal diagonal and $M(\boldsymbol{\alpha}_t) = \max_{i=1\ldots n} \{|\alpha_{i,t}|\}$, there is $0 < \delta_{n+1} \leq \delta'_{n+1}$ such that for any $(\mathbf{x}_t, \boldsymbol{\alpha}_t) \in B_{\delta_{n+1}}(\mathbf{x}^*) \times C_n$

$$\begin{aligned} \left\|\mathbf{x}^e_{t+1} - \mathbf{x}^*\right\| &\leq \|\mathbf{x}^e_t - \mathbf{x}^*\| + \|A_t\left(F(\mathbf{x}^e_t) - \mathbf{x}^e_t\right)\| \leq \\ &\leq \|\mathbf{x}^e_t - \mathbf{x}^*\| + M(\boldsymbol{\alpha}_t)\left\|(F(\mathbf{x}^e_t) - \mathbf{x}^e_t)\right\| < \delta_n \end{aligned} \tag{28}$$

This is due to the continuity of F, the fact that $F(\mathbf{x}^*) = \mathbf{x}^*$, and the inequality

$$M(\boldsymbol{\alpha}_t) \leq \max_{i=1\ldots n} \left\{\mu_{n+1} + |\alpha^*_i|\right\} = M_n(\boldsymbol{\alpha}^*)$$

following

$$|\alpha_i| = |\alpha_i - \alpha^*_i + \alpha^*_i| < |\alpha_i - \alpha^*_i| + |\alpha^*_i| \leq \mu_{n+1} + |\alpha^*_i|\,.$$

Now given $\delta_0, \mu_0 > 0$, thanks to (27) and (28), we can retrieve recursively $\{(\delta_n, \mu_n)\}$, $\{\delta_n\}$ being decreasing and $\{\mu_n\}$ increasing and diverging to $+\infty$, such that if $(\mathbf{x}_{t,}\boldsymbol{\alpha}_t) \in \left\{B_{\delta_{n+1}}(\mathbf{x}^*) \times C_n\right\}$ then $(\mathbf{x}_{t+1,}\boldsymbol{\alpha}_{t+1}) \in \left\{B_{\delta_n}(\mathbf{x}^*) \times B_{\mu_n}(\boldsymbol{\alpha}^*)\right\}$. So given $\boldsymbol{\alpha}$ we can find n such that $\boldsymbol{\alpha} \in C_n$ and, correspondingly, $\delta = \delta_{n+1}$ such that for any $\mathbf{x} \in B_\delta(\mathbf{x}^*)$ the system takes $(\mathbf{x}, \boldsymbol{\alpha})$ to $I_0(\mathbf{x}^*, \boldsymbol{\alpha}^*) = \left\{B_{\delta_0}(\mathbf{x}^*) \times B_{\mu_0}(\boldsymbol{\alpha}^*)\right\} \subset \mathfrak{B}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$, in finite steps (at most n). This concludes our proof.

■

Remark 3 *As in the unidimensional case, for the sake of proving Theorem 3, assumption [B3] can be slightly relaxed, in the following sense:*

[B4] $\forall \mu > 0\ \exists \gamma \in (0, 1) : \|\boldsymbol{\alpha}^* - G(\mathbf{x}^*, \boldsymbol{\alpha}_t)\| < \gamma \|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\| \quad \forall \boldsymbol{\alpha}_t \notin B_\mu(\boldsymbol{\alpha}^*)$, where $\|\cdot\|$ is the Euclidean norm. and $B_\mu(\boldsymbol{\alpha}^*)$ is a ball of radius μ centred in $\boldsymbol{\alpha}^*$.

Remark 4 *This way we can pick a $\hat{\mu}$ and a corresponding $\hat{\gamma}$ such that [B4] applies and, at the same time, $B_{\hat{\mu}}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ is strictly contained in $\mathfrak{B}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$; and this is sufficient to prove the theorem.*

We now establish, with the three following results, a link with Proposition 1. But first we need to state something in advance. We need to define a topology for the set of functions we are interested in, therefore we might make things easier by restricting to bounded functions or by considering a compact domain, and so resort to the τ_1 topology of C^1-uniform convergence. This is clearly a heavy requirement in terms of the economic interpretation. Hence, we will stick to a higher level of generality.

Let $\mathcal{F}$ be the set of C^1 functions defined on $\mathbb{R}^m$ taking values in $\mathbb{R}^n$. Consider, on this set, the following distance:

Definition 1 *Consider a "telescopic" sequence of compact sets* $K_i \subset K_{i+1}$ *such that* $\bigcup_i K_i = \mathbb{R}^m$. *On every compact* K_i *define*

$$\max_{x \in K_i} \left(\| f(x) \| + \| Df(x) \| \right)$$

where $\| f(x) \|$ *is the euclidean norm in* $\mathbb{R}^n$ *and* $\| Df(x) \|$ *is the norm the of the Jacobian* $Df(x)$ *defined by*

$$\| Df(x) \| = \max_{\|v\|=1} \| Df(x) v \|$$

Let, $d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$, *be the following function:*

$$d(f,g) = \lim_{i \to \infty} \frac{\max_{x \in K_i} \left(\| f(x) - g(x) \| + \| Df(x) - Dg(x) \| \right)}{1 + \max_{x \in K_i} \left(\| f(x) - g(x) \| + \| Df(x) - Dg(x) \| \right)}$$

Proposition 3 *The function* $d(f,g)$ *is a metric.*

Proof.

Clearly $d(f,g) \geq 0$ and $f \neq g$ implies that there is k_i such that

$$\max_{x \in K_i} \left(\| f(x) - g(x) \| + \| Df(x) - Dg(x) \| \right) > 0$$

implying $d(f,g) > 0$. Also, trivially, $d(f,g) = d(g,f)$.

Finally $d(f,g) \leq d(f,h) + d(h,g)$. Indeed, given a compact K_i, let $\tau_{1,i}$ be the metric of the C^1-uniform convergence for functions defined on K_i. Because $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \frac{t}{1+t}$ is increasing, and thanks to the triangular inequality

$$\begin{aligned}
\frac{\tau_{1,i}(f,g)}{1+\tau_{1,i}(f,g)} &\leq \frac{\tau_{1,i}(f,h) + \tau_{1,i}(h,g)}{1+\tau_{1,i}(f,h)+\tau_{1,i}(h,g)} = \\
&= \frac{\tau_{1,i}(f,h)}{1+\tau_{1,i}(f,h)+\tau_{1,i}(h,g)} + \frac{\tau_{1,i}(h,g)}{1+\tau_{1,i}(f,h)+\tau_{1,i}(h,g)} \leq \\
&\leq \frac{\tau_{1,i}(f,h)}{1+\tau_{1,i}(f,h)} + \frac{\tau_{1,i}(h,g)}{1+\tau_{1,i}(h,g)}
\end{aligned}$$

taking the limit for $i \to \infty$. ■

Remark 5 *Such distance is a form of* Poincaré metric *and we label it* τ_p.

Remark 6 *It's worth noting that* $\tau_p(f,g) \leq 1$ *for any* f,g *belonging to* $\mathcal{F}$.

Remark 7 *The metric* τ_p *on the set* $\mathcal{F}$ *does not come from any norm since*

$$|\lambda| \, \tau_p(f,0) \neq \tau_p(\lambda f, 0)$$

Proposition 4 *The distance τ_p does not depends on the choice of the telescopic sequence of compact sets.*

Proof.

Let B_j and K_i be two telescopic sequences of compact sets such that $\bigcup_j B_j = \bigcup_i K_i = \mathbb{R}^m$. Let d_B and d_K be the distances defined through $\{B_j\}$ and $\{K_i\}$ respectively. Suppose that f and g be such that $d_k(f,g) \neq d_B(f,g)$, and without loss of generality $d_k(f,g) < d_B(f,g)$; then for any $\varepsilon > 0$ there is $n(\varepsilon)$ such that

$$\left| \frac{\max_{x \in B_j} (\|f(x) - g(x)\| + \|Df(x) - Dg(x)\|)}{1 + \max_{x \in Bj} (\|f(x) - g(x)\| + \|Df(x) - Dg(x)\|)} - d_B(f,g) \right| < \varepsilon \quad (29)$$

for all $j > n(\varepsilon)$. Hence, there is $\bar{x}$ such that

$$\left| \frac{(\|f(\bar{x}) - g(\bar{x})\| + \|Df(\bar{x}) - Dg(\bar{x})\|)}{1 + (\|f(\bar{x}) - g(\bar{x})\| + \|Df(\bar{x}) - Dg(\bar{x})\|)} - d_B(f,g) \right| < \varepsilon$$

For such a $\bar{x}$ there is a K_{i_0} such that $\bar{x} \in K_{i_0}$, moreover

$$\frac{\max_{x \in K_{i_0}} (\|f(x) - g(x)\| + \|Df(x) - Dg(x)\|)}{1 + \max_{x \in K_{i_0}} (\|f(x) - g(x)\| + \|Df(x) - Dg(x)\|)} < d_K(f,g) \quad (30)$$

So, putting together (29) and (30), we have

$$d_K(f,g) > d_B(f,g) - \varepsilon$$

which contradict the assumption, as ε is arbitrarily small. ■

Remark 8 *Observe that*

$$\|f(x) - g(x)\| + \|Df(x) - Dg(x)\| < \varepsilon \qquad \text{for any } x \in \mathbb{R}^m$$

implies $\tau_p(f,g) < \varepsilon$ and that, on the contrary, if $\tau_p(f,g) < \sigma$ then for all $x \in \mathbb{R}^m$

$$\|f(x) - g(x)\| + \|Df(x) - Dg(x)\| < \frac{\sigma}{1 - \sigma}$$

which is smaller than 2σ for $\sigma < \frac{1}{2}$. In other words, two functions are near for the Poincaré metric *if and only if they are punctually ε-near everywhere.*

Notice that, with respect to Fuchs [8] and to assumptions [A1] and [B1] of Theorems 1 and 3, in the sequel we require that $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ be hyperbolic and attracting, not just asymptotically stable. It is not an overly strong restriction, as, given $\mathcal{F}_{\mathbf{y}}$, the space of bounded C^1 functions that have a fixed point in $\mathbf{y}$, and given on $\mathcal{F}_{\mathbf{y}}$ the topology τ_1, $\mathcal{F}_{\mathbf{y}}^{hyp}$ (the subset of $\mathcal{F}_{\mathbf{y}}$ for which y is hyperbolic) is open and dense in $\mathcal{F}_{\mathbf{y}}$.

It is straightforward to check that all the learning rules that satisfy the assumptions of Theorem 3 belong to the family $\mathcal{L}$ of Proposition 1:

Theorem 4 *Let* $\mathbf{x}^*$ *a locally unique, perfect foresight equilibrium for the map (2),* $\mathcal{E}(\mathbf{x}^*)$ *the set of expectations functions (4), parametrized by* $\boldsymbol{\alpha}$*, for which* $\mathbf{x}^*$ *is a temporary stationary attractive hyperbolic equilibrium. Let* $\mathcal{G}$ *the set of learning rules* $G(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)$ *satisfying [B1], [B2] and [B3]. Let* $\mathcal{L} \subseteq \mathcal{G}$ *be the set of learning rules such that, given the map*

$$(M): \begin{cases} \mathbf{x}_{t+1}^e = E(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) = \mathbf{x}_t^e + A_t [F(\mathbf{x}_t^e) - \mathbf{x}_t^e] \\ \boldsymbol{\alpha}_{t+1} = G(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) \end{cases}$$

there is a neighbourhood U *of* $\mathbf{x}^*$ *such that:*

a) the trajectories $\{\mathbf{y}_t\} = \{\mathbf{x}_t, \boldsymbol{\alpha}_t\}$ *determined by the initial condition* $\mathbf{y}_0 = (\mathbf{x}_0, \boldsymbol{\alpha}_0)$ *with* $\mathbf{x}_0$ *in* U *and* $\boldsymbol{\alpha}_0$ *in* $\mathcal{E}(\mathbf{x}^*)$*, converge to* $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ *with* $\boldsymbol{\alpha}^* \in \mathcal{E}(\mathbf{x}^*)$*;*

b) there exist trajectories $\{\mathbf{y}_t\}$ *determined by initial conditions* $\mathbf{y}_0 = (\mathbf{x}_0, \boldsymbol{\alpha}_0)$ *with* $\mathbf{x}_0$ *in* U *and* $\boldsymbol{\alpha}_0 \notin \mathcal{E}(\mathbf{x}^*)$*, that still converge to* $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ *with* $\boldsymbol{\alpha}^* \in \mathcal{E}(\mathbf{x}^*)$*.*

Then $\mathcal{L} = \mathcal{G}$*.*

Proof.

If $\mathcal{E}(\mathbf{x}^*) = \mathcal{A}$ the problem is irrelevant because all initial conditions have $\boldsymbol{\alpha}_0 \in \mathcal{E}(\mathbf{x}^*)$. If, vice versa, there is $\boldsymbol{\alpha}$ belonging to $\mathcal{A}$ but not to $\mathcal{E}(\mathbf{x}^*)$: by Theorem 3, given $G \in \mathcal{G}$ and $\boldsymbol{\alpha}$, we can find $\delta > 0$ (depending on $\boldsymbol{\alpha}$) such that the trajectories $\{\mathbf{y}_t\}$ following the initial condition $\mathbf{y}_0 = (\mathbf{x}_0, \boldsymbol{\alpha}_0)$ with $\mathbf{x}_0$ belonging to a neighbourhood U of $\mathbf{x}^*$ and radius δ and $\boldsymbol{\alpha}_0$ belonging to the closure of a neighbourhood I of $\boldsymbol{\alpha}^*$ of radius $\|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}\|$, converge to $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ with $\boldsymbol{\alpha}^* \in \mathcal{E}(\mathbf{x}^*)$. So $G \in \mathcal{G} \Rightarrow G \in \mathcal{L}$, therefore $\mathcal{G} \subseteq \mathcal{L}$ while by definition $\mathcal{L} \subseteq \mathcal{G}$.

■

Hence, if we substitute the assumption (1) of Fuchs with [A4] or [B4], the negative results in [8] is completely reversed.

In fact we can say even more. As we anticipated in Section 3 the subset of learning rules we are considering is not a “thin set”, in a topological sense. This is what we are going to show next, in Theorem 2.

First, we need the following facts.

Lemma 2 *Let* $f \in \mathcal{F}$*. The following statements are equivalent:*

a) f *is a contraction of constant* γ*, i.e.*

$$d(f(x), f(y)) \leq \gamma d(x, y)$$

(d is the euclidean distance)

b) f *is such that*

$$\|Df(x)\| < \gamma \qquad \text{for any } x \in \mathbb{R}^n$$

Proof. (a $\Rightarrow$ b) :

Let f a contraction of constant γ, we must show that $\|Df(x)v\| < \gamma$ for all $\|v\| = 1$. The inequality

$$\|Df(x)v\| = \left\|\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t}\right\| < \lim_{t\to 0}\frac{\gamma\|tv\|}{|t|} = \gamma$$

proves this.

(b $\Rightarrow$ a) :

Let us estimate $\|f(x) - f(y)\|$. We have:

$$f(y) - f(x) = \int_0^1 \frac{d}{dt} f(x + t(y-x))\,dt = \int_0^1 Df(x+t(y-x))(y-x)\,dt$$

hence

$$\|f(x) - f(y)\| \le \int_0^1 \|Df(x+t(y-x))\|\,\|y-x\|\,dt < \int_0^1 \gamma\|y-x\|\,dt = \gamma\|y-x\|$$

■

Lemma 3 *Let $f \in \mathcal{F}$ a contraction of constant γ. For any ε such that $\gamma + \varepsilon < 1$, there is δ_ε such that*

$$\tau_p(f,g) < \delta_\varepsilon \Longrightarrow g \text{ is a contraction of constant } \beta < \gamma + \varepsilon$$

Proof. Let f be a contraction of constant γ and $g \in \mathcal{F}$ such that $\tau_p(f,g) < \varepsilon$. From the definition of τ_p it follows that for any $x \in \mathbb{R}^n$ (see Remark 8)

$$\|Df(x) - Dg(x)\| < \frac{\varepsilon}{1-\varepsilon} < 2\varepsilon \qquad \text{if} \qquad \varepsilon < \frac{1}{2}$$

Therefore for $\varepsilon < \frac{1}{2}$

$$|\|Df(x)\| - \|Dg(x)\|| < \|Df(x) - Dg(x)\| < 2\varepsilon$$

so

$$\|Dg(x)\| < \|Df(x)\| + 2\varepsilon$$

By Lemma 2, given ε such that $\gamma + 2\varepsilon < 1$, it follows that g is a contraction, as required. ■

Lemma 4 *Let $f \in \mathcal{F}$ be a contraction of constant γ, $z^* = f(z^*)$ and $B_\eta(z^*)$ an η-ball centred in z^*: then $f(B_\eta(z^*)) \subseteq B_{\eta\gamma}(z^*)$.*

Proof. By definition, for all $z \in B_\eta(z^*)$ we have

$$\|f(z) - z^*\| = \|f(z) - f(z^*)\| \le \gamma \|z - z^*\| \le \gamma \max_{y \in B_\eta(z^*)} \|y - z^*\| = \eta\gamma$$

■

Theorem 5 *Let $\mathbf{x}^*$ a locally unique, perfect foresight stationary equilibrium for the map (2), $\mathcal{E}(\mathbf{x}^*)$ the set of expectations functions (4), parametrized by $\boldsymbol{\alpha}$, for which $\mathbf{x}^*$ is a temporary stationary attractive hyperbolic equilibrium. Let $\mathcal{G}$ be the set of learning rules $G(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)$ for which the system (M) satisfies [B1] to [B3]. Consider the set $\Gamma = \{G : \boldsymbol{\alpha}_{t+1} = G(\mathbf{x}_t^e, \boldsymbol{\alpha}_t); G \in C^1\}$, of the C^1 learning rules, endowed with the topology τ_p.*

Then $\mathcal{G}$, as a subset of Γ, has a non empty interior.

Proof.

To show that the set is not empty consider

$$\hat{G}(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) = \boldsymbol{\alpha}^* + \gamma_1 (\boldsymbol{\alpha}_t - \boldsymbol{\alpha}^*)$$

with $\boldsymbol{\alpha}^*$ in $\mathcal{E}(\mathbf{x}^*)$ and $\gamma_1 \in (0,1)$. It is easy to see that [B1], [B2] and [B3] (the latter for any $\gamma_1 < \gamma < 1$) apply to the corresponding (M) and so $\hat{G} \in \mathcal{G}$.

Notice that $\hat{G}$ is a contraction:

$$\begin{aligned}\left\|\hat{G}(\mathbf{x}_1, \boldsymbol{\alpha}_1) - \hat{G}(\mathbf{x}_2, \boldsymbol{\alpha}_2)\right\| &= \|[\boldsymbol{\alpha}^* + \gamma_1(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}^*)] - [\boldsymbol{\alpha}^* + \gamma_1(\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}^*)]\| = \\ &= \|\gamma_1(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\| = |\gamma_1| \|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\| \le |\gamma_1| \|(\mathbf{x}_1, \boldsymbol{\alpha}_1) - (\mathbf{x}_2, \boldsymbol{\alpha}_2)\|\end{aligned}$$

Now, consider a contraction of constant γ, $G(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) : \mathbb{R}^{2n} \to \mathbb{R}^n$; thanks to Lemma 3 there is an ε-neighbourhood, J, of G, (in the τ_p topology) entirely made up of contractions. We want to show that every such contraction verifies [B1], [B2] and [B3]. For any $G^j \in J$ its restriction $G^j_{\mathbf{x}^*}(\boldsymbol{\alpha}_t)$ is again a contraction of $\mathbb{R}^n$ in itself. As a consequence of Lemma 4 and Remark 8, for each $\eta > 0$ such that

$$\eta(1-\gamma) > \frac{\varepsilon}{1-\varepsilon}$$

we have

$$G^j_{\mathbf{x}^*}(B_\eta(\boldsymbol{\alpha}^*)) \subseteq B_\eta(\boldsymbol{\alpha}^*) \tag{31}$$

(31) and the Contraction Mapping Theorem imply that $G^j_{\mathbf{x}^*}$ has a fixed point, $\boldsymbol{\alpha}^{*j}$, in $\overline{B_\eta(\boldsymbol{\alpha}^*)}$, hence $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ is a fixed point for the system

$$(M^j) : \begin{cases} \mathbf{x}_{t+1}^e = E(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) = \mathbf{x}_t^e + A_t[F(\mathbf{x}_t^e) - \mathbf{x}_t^e] \\ \boldsymbol{\alpha}_{t+1} = G^j(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) \end{cases}$$

So, [B2] is obvious, and [B3] immediately follows the fact that G^j is a contraction. In so far as it concerns the [B1] assumption consider the Jacobian of M^j evaluated in $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$

$$M^{j'}(\mathbf{x}^*, \boldsymbol{\alpha}^*) = \begin{pmatrix} E'(\mathbf{x}^*, \boldsymbol{\alpha}^*) & 0 \\ G^j_{\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\alpha}^*) & G^j_{\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}^*) \end{pmatrix}$$

where E' is the Jacobian for the system (4), whereas $G^j_{\mathbf{x}}$ and $G^j_{\boldsymbol{\alpha}}$ contain the derivatives of G^j with respect to $\mathbf{x}$ and (respectively) $\boldsymbol{\alpha}$. The hyperbolicity conditions on the eigenvalues are granted for a small enough ε. ■

Remark 9 *The above results are not heavily influenced by the chosen topology. Restricting the analysis to the set (endowed with the τ_1 topology of the C^1-uniform convergence) of functions defined on compacts and continuously differentiable, Theorem 5 would still work and the proof could easily be adapted.*

Now to the last step. We now show that every ELR with the property (1) can be approximated by a $G \in \mathcal{G}$ with arbitrary precision.

Theorem 6 *Let $\mathbf{x}^*$ a locally unique, perfect foresight stationary equilibrium for the map (2), $\mathcal{E}(\mathbf{x}^*)$ the set of expectations functions (4), parametrized by $\boldsymbol{\alpha}$, for which $\mathbf{x}^*$ is a temporary stationary attractive hyperbolic equilibrium. Let $\mathcal{G}$ be the set of learning rules $G(\mathbf{x}^e_t, \boldsymbol{\alpha}_t)$ for which the system (M) satisfies [B1] to [B3]. Consider the set $\Gamma = \{G : \boldsymbol{\alpha}_{t+1} = G(\mathbf{x}^e_t, \boldsymbol{\alpha}_t); G \in C^1\}$, of the C^1 learning rules, endowed with the topology τ_p and $\mathcal{G}_F \subset \Gamma$ the set of functions $G_F(\mathbf{x}^e_t, \boldsymbol{\alpha}_t)$ for which $G_F(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) = \boldsymbol{\alpha}_t$. Then, for any $G_F \in \mathcal{G}_F \subset \Gamma$ there is a sequence $\{G_n\}$, with $G_n \in \mathcal{G} \subset \Gamma$ for all n, such that*

$$\lim_{n\to+\infty} G_n(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) = G_F(\mathbf{x}^e_t, \boldsymbol{\alpha}_t) \qquad \text{for any } (\mathbf{x}^e_t, \boldsymbol{\alpha}_t)$$

Proof.

Consider $\boldsymbol{\alpha}_{t+1} = G_F(\mathbf{x}^*, \boldsymbol{\alpha}_t) \in \mathcal{G}_F$. Let $\boldsymbol{\alpha}^* \in \mathcal{E}(\mathbf{x}^*)$. Set

$$G_n(\mathbf{x}^*, \boldsymbol{\alpha}_t) = \frac{1}{n}\boldsymbol{\alpha}^* + \left(1 - \frac{1}{n}\right) G_F(\mathbf{x}^*, \boldsymbol{\alpha}_t)$$

Clearly the sequence converges pointwise to G_F. We show that $G_n \in \mathcal{G}$ for all $n \in \mathbb{N}$:

- As for [B1] we have

$$G_n(\mathbf{x}^*, \boldsymbol{\alpha}^*) = \frac{1}{n}\boldsymbol{\alpha}^* + \left(1 - \frac{1}{n}\right) G_F(\mathbf{x}^*, \boldsymbol{\alpha}^*) = \frac{1}{n}\boldsymbol{\alpha}^* + \left(1 - \frac{1}{n}\right)\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^*$$

so $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ is a fixed point for

$$(M_n) : \begin{cases} \mathbf{x}_{t+1}^e = E(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) = \mathbf{x}_t^e + A_t [F(\mathbf{x}_t^e) - \mathbf{x}_t^e] \\ \boldsymbol{\alpha}_{t+1} = G_n(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) \end{cases}$$

Moreover consider $G_{n,\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$, i.e. the matrix of partial derivatives of G_n with respect to $\boldsymbol{\alpha}$; then

$$G_{n,\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}^*) = \left(1 - \frac{1}{n}\right) G_{F,\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$$

As $G_{F,\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}_t) = \boldsymbol{\alpha}_t$ for any $\boldsymbol{\alpha}_t$, $G_{F,\boldsymbol{\alpha}}(\mathbf{x}^*, \cdot)$ coincides with the identity matrix and so $G_{n,\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ has eigenvalues equal to $\left(1 - \frac{1}{n}\right)$. Then, provided that $\boldsymbol{\alpha}^* \in \mathcal{E}(\mathbf{x}^*)$, $(\mathbf{x}^*, \boldsymbol{\alpha}^*)$ is hyperbolic and attracting for the map (M_n).

B2 trivially holds

- Finally remark that:

$$\begin{aligned} \|\boldsymbol{\alpha}^* - G_n(\mathbf{x}^*, \boldsymbol{\alpha}_t)\| &= \left\|\boldsymbol{\alpha}^* - \frac{1}{n}\boldsymbol{\alpha}^* - \left(1 - \frac{1}{n}\right) G_F(\mathbf{x}^*, \boldsymbol{\alpha}_t)\right\| = \\ &= \left\|\left(1 - \frac{1}{n}\right)\boldsymbol{\alpha}^* - \left(1 - \frac{1}{n}\right)\boldsymbol{\alpha}_t\right\| = \left(1 - \frac{1}{n}\right)\|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\| < \\ &< \gamma \|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\| \end{aligned}$$

  having set $\left(1 - \frac{1}{n}\right) < \gamma < 1$.

This completes our proof. ■

Remark 10 *Theorem 6 only provides pointwise convergence. This limitation cannot be easily removed endowing* Γ *of a different metric: in fact, given a function in* $\mathcal{G}_F$*, a function in* $\mathcal{G}$ *and a point* $(\mathbf{x}^*, \boldsymbol{\alpha}_t)$

$$\|G_F(\mathbf{x}^*, \boldsymbol{\alpha}_t) - G(\mathbf{x}^*, \boldsymbol{\alpha}_t)\| = \|\boldsymbol{\alpha}_t - G(\mathbf{x}^*, \boldsymbol{\alpha}_t)\| \geq (1 - \gamma)\|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\|$$

so that we can make this distance as great as we want if $\|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_t\|$ *(i.e. the domain) is unbounded.*

Remark 11 *If we consider only the set of* C^1 *functions, defined on a compact set, endowed with the metric* τ_1 *of* C^1*-uniform convergence, the theorem 6 can be proved changing the thesis in:*

[...Then, for all $G_F \in \mathcal{G}_F \subset \Gamma$ there is a sequence $\{G_n\}$, with $G_n \in \mathcal{G} \subset \Gamma$ for all n, such that:

$$\lim_{n \to +\infty} G_n = G_F$$

...].

The proof remains unchanged in the existing part. It remains to be proved that the sequence uniformly converges to G_F. Indeed we have:

$$\tau_1 (G_n, G_F) = \max_{(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) \in K} (\|G_n - G_F\| + \|DG_n - DG_F\|)$$

where $\|G(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)\|$ is the euclidean norm in $\mathbb{R}^n$ and $\|DG(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)\|$ is the norm of the Jacobian matrix $DG(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)$ defined by

$$\|DG(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)\| = \max_{\|v\|=1} \|DG(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) v\|$$

But

$$\|G_n - G_F\| = \frac{1}{n} \|\boldsymbol{\alpha}^* - G_F(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)\|$$

and

$$\|DG_n - DG_F\| = \|D(G_n - G_F)\| = \frac{1}{n} \|D(\boldsymbol{\alpha}^* - G_F(\mathbf{x}_t^e, \boldsymbol{\alpha}_t))\|$$

hence

$$\tau_1 (G_n, G_F) = \frac{1}{n} \max_{(\mathbf{x}_t^e, \boldsymbol{\alpha}_t) \in K} (\|\boldsymbol{\alpha}^* - G_F(\mathbf{x}_t^e, \boldsymbol{\alpha}_t)\| + \|D(\boldsymbol{\alpha}^* - G_F(\mathbf{x}_t^e, \boldsymbol{\alpha}_t))\|)$$

and, because both norms are bounded, for any $\varepsilon > 0$ there is n_ε such that for all $n > n_\varepsilon$ it is

$$\tau_1 (G_n, G_F) < \varepsilon$$

which proves what stated.

5 Conclusions

In this paper we dealt with a general dynamic model, potentially consistent with a wide class of economic models under bounded rationality in which the state variable depends on agent's predictions about the future and featuring evolution of the expectation function through which traders perform their forecasts. This idea basically aims at overcoming some of the limits flawing the model with given (adaptive) expectation function, namely the existence, when the parameter is above a certain bifurcation value, of non-perfect foresight attracting cycles. This implies that a form of systematic error will characterize the limiting behaviour of the economy. Moreover, there is convergence towards these cycles even if the system starts arbitrarily close to the steady state.

Our analysis, performed under the assumption that the expectation function is chosen in a set whose elements are parametrized by the learning step coefficient, α, indicates that, from a local point of view, the stability conditions are more restrictive in a way that typically depends on the shape of the function used by the agents to update α. On the other hand, global analysis suggests that, if we have a stable steady state, whatever the initial value on α there is an open interval around the equilibrium point for the state variable x, belonging to the basin of attraction of the fixed point. This result is valuable because it enlarges the domain of "good" parameter values, that is compatible with convergence to a situation of self-fulfilling expectations.

We compared the outcome with the existing literature, in particular with Fuchs [8], in which it was shown that, ascribing to the agents the possibility of choosing among alternatives the mechanism generating their forecasts, the system's stability is enhanced in a negligible set of cases. This conclusions draws heavily on assumptions which we showed to be verified only in a "marginal" set of, economically consistent, models.

The present work put forward several indications for further research: the analysis outlined here can be extended to cover more complex dynamic behaviour, such as cycles and other attractors; also, many economic models treated in the bounded rationality literature may be the subject of application of the results of this paper, and numerical simulations could be performed to sketch, for instance, the actual shape of the basins of attraction.

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